

THE COHOMOLOGICAL HALL ALGEBRA OF A PREPROJECTIVE ALGEBRA

YAPING YANG AND GUFANG ZHAO

Dedicated to Professor Jerzy Weyman on the occasion of his 60th birthday.

ABSTRACT. We introduce for each quiver Q and each algebraic oriented cohomology theory A , the cohomological Hall algebra (CoHA) of Q , as the A -homology of the moduli of representations of the preprojective algebra of Q . This generalizes the K -theoretic Hall algebra of commuting varieties defined by Schiffmann-Vasserot [SV12]. We construct an action of the preprojective CoHA on the A -homology of Nakajima quiver varieties. We compare this with the action of the Borel subalgebra of Yangian when A is the intersection theory. We also give a shuffle algebra description of this CoHA in terms of the underlying formal group law of A . As applications, we obtain a shuffle description of the Yangians.

CONTENTS

0. Introduction	1
1. Algebraic oriented cohomology theory	7
2. The formal cohomological Hall algebras	14
3. The generalized shuffle algebras	16
4. The preprojective cohomological Hall algebras	21
5. Representations of the preprojective CoHA	23
6. Yangians and shuffle algebras	33
7. Comparison with Yangian actions on quiver varieties	36
Appendix A. A symmetric polynomial identity	41
References	43

0. INTRODUCTION

Let Q be a quiver, and \mathfrak{g}_Q the corresponding symmetric Kac-Moody Lie algebra. The Nakajima quiver varieties of Q are fine moduli spaces parameterizing stable framed representations of the preprojective algebra of Q [Nak94]. They play an important role in constructing representations of various quantum groups associated to \mathfrak{g}_Q .

When Q is a quiver without edge loops, Nakajima constructed an action of the quantum loop algebra $U_q(L\mathfrak{g}_Q)$ on the equivariant K -theory of quiver varieties [Nak01]. He realized the Drinfeld generators of $U_q(L\mathfrak{g}_Q)$ as explicit convolution operators, and showed that they satisfy the required commutation relations. Following a similar method, Varagnolo constructed an action of the Yangian $Y_h(\mathfrak{g}_Q)$ on the equivariant Borel-Moore homology of quiver varieties [Va00].

Date: April 8, 2016.

2010 Mathematics Subject Classification. Primary 17B37; Secondary 14F43, 55N22.

Key words and phrases. Oriented cohomology theory, quiver variety, Hall algebra, Yangian, shuffle algebra.

The main goal of the present paper is to show that for any quiver Q and an arbitrary algebraic oriented cohomology theory (OCT) A in the sense of Levine-Morel [LM07], there is an affine quantum group associated to \mathfrak{g}_Q and A , which acts on the A -homology of quiver varieties. Note that the method in [Nak01, Va00] is useful in constructing representations of algebras with known generators and relations. However, it does not provide a construction of the algebras themselves, i.e., the operators used in the action have no prescribed commutation relations.

The existence of such a quantum group is predicted by the parallelism between three different mathematical objects [GKV95]: affine quantum groups, oriented cohomology theories, and formal group laws. The correspondence between the last two is well-known. For the formal group laws coming from 1-dimensional algebraic groups, the corresponding affine quantum groups are known. However, no construction of affine quantum groups from other formal group laws has been available. The present paper gives the first construction of quantum groups in the affine setting beyond the Yangians, quantum loop algebras, and elliptic quantum groups.

We construct an algebra, called the preprojective cohomological Hall algebra (CoHA), as well as an action of it on the A -homology of Nakajima quiver varieties associated to Q , following an approach similar to [SV12]. The Nakajima-type raising operators are lifted as elements in this CoHA. We also construct a suitable extension of the preprojective CoHA, a subalgebra of which is a quantization of $U(\mathfrak{b}_Q[[u]])$, where the quantization depends on the underlying formal group law of A . We study in details the test case when the formal group law is additive, and we show in this case the quantization is the Borel subalgebra of the Yangian $Y_h(\mathfrak{g}_Q)$.

To describe the multiplication of the preprojective CoHA algebraically, we also give a shuffle formula. Historically, a shuffle description was not only used as an intrinsic way of defining quantum groups [R98], but also provided interesting combinatorial information. For example, PBW property of quantum groups becomes transparent through the shuffle algebra (see, e.g., [R89]), and the dual canonical basis has a purely algebraic description in terms of it [Lec04]. In this paper, we give a geometric description of the shuffle algebra. The multiplication of Nakajima-type raising operators can be calculated using the shuffle formula.

The constructions in the present paper follows similar idea as [SV12]. When Q is the Jordan quiver, the Nakajima quiver variety is isomorphic to $\text{Hilb}^n(\mathbb{A}^2)$, the Hilbert scheme of n points on \mathbb{A}^2 . Schiffmann-Vasserot constructed an action of the elliptic Hall algebra on the equivariant K -theory of $\coprod_{n \in \mathbb{N}} \text{Hilb}^n(\mathbb{A}^2)$ [SV12]. Feigin-Tsybaliuk independently constructed an action of the Feigin-Odesskii shuffle algebra on the equivariant K -theory of $\coprod_{n \in \mathbb{N}} \text{Hilb}^n(\mathbb{A}^2)$ [FT09]. A homomorphism from the elliptic Hall algebra to the shuffle algebra is further given in [SV12], which is compatible with their actions. The machinery in the present paper, applied to the special case when A is the Chow group, and Q a quiver without edge-loops, gives a new construction of the Yangian, in terms of cohomological Hall algebra. This is an affine analogue of the classical theorem that the (extended) Ringel-Hall algebra of Q is isomorphic to $U_q(\mathfrak{b}_Q)$ [Rin90]. Although this special case might be expected by analogy with [SV12], no precise statement or proof had been given. As a corollary, we provide a shuffle description of the Yangian, which was not known previously, though a similar description for the quantum loop algebra $U_q^+(L\mathfrak{g}_Q)$ has been known in various special cases (see [En00] for the case when \mathfrak{g} is of finite type which is not G_2 , [SV12] when $\mathfrak{g} = \widehat{\mathfrak{gl}}_1$, and [Ne15] when \mathfrak{g} is $\widehat{\mathfrak{sl}}_n$).

The machinery developed in the present paper will be applied further. In a sequel [YZ1], we will show that the extended preprojective CoHA is a bialgebra, whose (reduced) Drinfeld double is a quantization of $U(\mathfrak{g}_Q[[u]])$. In the case when A is the Chow ring, we will construct the double Yangian as a suitable localization of the Drinfeld double of the extended preprojective CoHA. In

another sequel [YZ2], we will study the case when A is the equivariant elliptic cohomology. The corresponding affine quantum group is the Felder's elliptic quantum group [Fed94].

0.1. The preprojective CoHA. Let $Q = (I, H)$ be a quiver, with set of vertices I and arrows H . Let $\text{Rep}(Q, v)$ be the affine variety parametrizing representations of Q with dimension vector $v = (v^i)_{i \in I} \in \mathbb{N}^I$. The vector space $\text{Rep}(Q, v)$ carries a natural action of $G_v = \prod_{i \in I} \text{GL}_{v^i}$. Let $\mu_v : T^* \text{Rep}(Q, v) \rightarrow \mathfrak{gl}_v^*$ be the moment map of the cotangent bundle of $\text{Rep}(Q, v)$. The torus $T = \mathbb{G}_m^2$ acts on $T^* \text{Rep}(Q, v)$ with the first \mathbb{G}_m -factor scaling $\text{Rep}(Q, v)$ and the second one scaling the fibers of $T^* \text{Rep}(Q, v)$ (see Assumption 3.1 for the conditions on this action). We set

$${}^A\mathcal{P}(Q) := \bigoplus_{v \in \mathbb{N}^I} {}^A\mathcal{P}_v(Q) = \bigoplus_{v \in \mathbb{N}^I} A_{T \times G_v}(\mu_v^{-1}(0)).$$

We will use the abbreviations \mathcal{P} and \mathcal{P}_v if both A and Q are understood from the context. In § 4, we define maps

$$m_{v_1, v_2}^P : \mathcal{P}_{v_1} \otimes \mathcal{P}_{v_2} \rightarrow \mathcal{P}_{v_1 + v_2}.$$

Theorem A (Theorem 4.1). The \mathbb{N}^I -graded abelian group \mathcal{P} , endowed with m_{v_1, v_2}^P , is an associative, \mathbb{N}^I -graded algebra over $A_T(\text{pt}) \cong A(\text{pt})[[t_1, t_2]]$.

This algebra will be called the *preprojective cohomological Hall algebra* (preprojective CoHA). The name is motivated by the fact that the subvariety $\mu_v^{-1}(0) \subset T^* \text{Rep}(Q, v)$ parameterizes the representations of the preprojective algebra of Q . Recall that the latter is the quotient of the path algebra of the doubled quiver \bar{Q} by the relations $\sum_{x \in H} [x, x^*] = 0$, where for any arrow x in Q the opposite arrow in \bar{Q} is denoted by x^* .

Our construction of the algebra \mathcal{P} has two sources. One of them is the construction of a K -theoretic Hall algebra of commuting varieties defined by Schiffmann-Vasserot in [SV12], the idea of which can be traced back to Grojnowski [Gr94b]. Our construction for an arbitrary quiver and OCT essentially follows from the same techniques. Another source is the construction of Kontsevich-Soibelman of a Hall multiplication on the critical cohomology (cohomology valued in a vanishing cycle) of representation spaces of a quiver with potential, which arises from the study of Donaldson-Thomas invariants (DT for short) [KoSo11]. Our preprojective CoHA can be considered as CoHA for the 2-Calabi-Yau category of representations of the preprojective algebra.

0.2. Action on quiver varieties. Let $\mathfrak{M}(v, w)$ be the Nakajima quiver variety with dimension vectors $v, w \in \mathbb{N}^I$ and stability condition θ^+ (see § 5.1). We construct a representation of the algebra ${}^A\mathcal{P}(Q)$ on the A -homology of the Nakajima quiver varieties

$${}^A\mathcal{M}(w) := \bigoplus_{v \in \mathbb{N}^I} A_{G_w \times T}(\mathfrak{M}(v, w)).$$

Theorem B (Theorem 5.4). For any $w \in \mathbb{N}^I$, there is a homomorphism of \mathbb{N}^I -graded $A_T(\text{pt})$ -algebras

$$a^{\text{prepr}} : {}^A\mathcal{P} \rightarrow \text{End}({}^A\mathcal{M}(w)).$$

For each $k \in I$, let e_k be the dimension vector valued 1 at vertex k and zero otherwise. We define the *spherical preprojective CoHA* to be the subalgebra ${}^A\mathcal{P}^s(Q) \subseteq {}^A\mathcal{P}(Q)$ generated by $\mathcal{P}_{e_k} = A_{T \times G_{e_k}}(\mu_{e_k}^{-1}(0))$ as k varies in I .

The map a^{prepr} , when restricted to the spherical subalgebra \mathcal{P}^s , factors through the action of the convolution algebra of the Steinberg variety $Z = \mathfrak{M}(w) \times_{\mathfrak{M}_0(w)} \mathfrak{M}(w)$ on ${}^A\mathcal{M}(w)$. In $A_{G_w \times \mathbb{G}_m}(Z)$ there are Nakajima-type operators, which are cohomology classes on Hecke correspondences. In

Theorem 5.6, we write down explicit elements in \mathcal{P}^s which map to the Nakajima-type operators under a^{repr} . Hence, for a general quiver, \mathcal{P}^e is a bigger symmetry on $\mathcal{M}(w)$ than that of [Nak01, Va00]. We expect the representations in Theorem B to produce highest weight integrable representations of the Drinfeld double of $\mathcal{P}^{s,e}$.

0.3. The shuffle description. Let ${}^A\mathcal{SH}$ be the shuffle algebra associated to Q and the formal group law of A (see § 3). It is a modified version of the Feigin-Odesskii shuffle algebra [FO97, FT09]. In § 3, we give the definition and a geometric interpretation of ${}^A\mathcal{SH}$, and hence prove the following.

Theorem C (Theorem 4.3). There is an algebra homomorphism $\Theta : {}^A\mathcal{P} \rightarrow {}^A\mathcal{SH}$. This algebra homomorphism becomes an isomorphism after localization at the augmentation ideal in $A_T(\text{pt})$.

The homomorphism Θ gives an explicit description of the Hall multiplication of \mathcal{P} using the shuffle formulas. Moreover, the image of ${}^A\mathcal{P}^s(Q)$ in \mathcal{SH} is a deformation of $U(\mathfrak{n}_Q[[u]]) \subseteq U(\mathfrak{g}_Q[[u]])$ associated to the formal group law of A . In § 5.4, we construct a commutative algebra ${}^A\mathcal{P}^0$, which acts on ${}^A\mathcal{P}$. We define the extended spherical CoHA to be $\mathcal{P}^{s,e} := \mathcal{P}^s \rtimes \mathcal{P}^0$, which is a quantization of $U(\mathfrak{b}_Q[[u]])$. In an up-coming publication [YZ1], we will define a comultiplication on $\mathcal{P}^{s,e}$, making it a bialgebra, whose Drinfeld double is a quantization of $U(\mathfrak{g}_Q[[u]])$. These quantum groups should have equally interesting applications, some perspectives are sketched in [YZ1, Example 3.9].

0.4. The Yangians $Y_h(\mathfrak{g}_Q)$ and Y_{MO} . We compare the preprojective CoHA with known quantum groups. In order to do so, we need to twist the multiplication of \mathcal{P} by a sign. More precisely, let $\tilde{\mathcal{P}}$ be the twisted preprojective CoHA, whose underlying abelian group is the same as \mathcal{P} , and the multiplication $m_{v_1, v_2}^{\tilde{\mathcal{P}}}$ on $\tilde{\mathcal{P}}$ differs from $m_{v_1, v_2}^{\mathcal{P}}$ by a sign coming from the Euler-Ringel form, spelled out in § 5.6. Similar to the untwisted case, we have the spherical subalgebra $\tilde{\mathcal{P}}^s \subseteq \tilde{\mathcal{P}}$. Define $\tilde{\mathcal{P}}^s$ to be the quotient of $\tilde{\mathcal{P}}^s|_{t_1=t_2=\hbar/2}$ by the \hbar -torsion part. Let $\tilde{\mathcal{P}}^{s,e}$ be $\tilde{\mathcal{P}}^s \rtimes \mathcal{P}^0$.

Now let A be the intersection theory CH (that is, the Chow group, see [Ful84]). Following Nakajima [Nak01], Varagnolo constructed an action of the Yangian $Y_h(\mathfrak{g}_Q)$ on the equivariant Chow groups of quiver varieties [Va00]. From entirely different considerations, Maulik-Okounkov constructed another algebra, Y_{MO} which also acts on the equivariant Chow groups of quiver varieties [Va00].

It is stated in [MO12] that $Y_h(\mathfrak{g}_Q)$ embeds into Y_{MO} , and that this embedding is compatible with their actions on $\mathcal{M}(w)$. However, there was no complete proof in the literature so far. (See [M13, § 6] for a partial result.) We spell out a proof of this fact in § 7.

Theorem D (Theorems 7.3 and 7.7). Let Q be a quiver without edge loops. Let $Y_h^{\geq 0}(\mathfrak{g}_Q)$ be the Borel subalgebra of $Y_h(\mathfrak{g}_Q)$.

- (1) There is an injective algebra homomorphism ${}^{\text{CH}}\tilde{\mathcal{P}}^{s,e} \hookrightarrow Y_{\text{MO}}^{\geq 0}$, whose image is the subalgebra $Y_h^{\geq 0}(\mathfrak{g}_Q) \subseteq Y_{\text{MO}}^{\geq 0}$.
- (2) For any $w \in \mathbb{N}^I$, let $\mathcal{M}(w) = \bigoplus_{v \in \mathbb{N}^I} \text{CH}_{G_w \times T}(\mathfrak{M}(v, w))$. The embedding in (1) makes the following diagram commutative

$$\begin{array}{ccc} {}^{\text{CH}}\tilde{\mathcal{P}}^{s,e} & \xrightarrow{\cong} & Y_h^{\geq 0}(\mathfrak{g}_Q) \\ \downarrow & & \downarrow \\ \text{End}(\mathcal{M}(w)) & \longleftarrow & Y_h\mathfrak{g}_Q \end{array}$$

where the map $Y_h\mathfrak{g}_Q \rightarrow \text{End}(\mathcal{M}(w))$ is defined in [Va00].

Theorem D can be thought of as an affine analogue of Ringel's Hall algebra description of the Borel of the quantum enveloping algebra of \mathfrak{g}_Q [Rin90]. In the affine setting, there are three quantized enveloping algebras: the Yangian, the affine quantum group, and the elliptic quantum group. A Hall algebra description of the positive part of the affine quantum group is a special case of Lusztig's construction of the composition subalgebra [L91]. However, a Hall algebra description of the Yangian was not known.

Combining Theorem D with Theorem C shows that the positive part of the Yangian $Y_h^+(\mathfrak{g}_Q)$ embeds into the shuffle algebra \mathcal{SH} . This can be considered as an affine analogue of [R98], that $U_q^+(\mathfrak{g}_Q)$ is a subalgebra of the quantum shuffle algebra. Although the shuffle description for the quantum loop algebra $U_q^+(L\mathfrak{g})$ has been known in various special cases (see [En00] for the case when \mathfrak{g} is of finite type which is not G_2 , and [Ne15] for the case when $\mathfrak{g} = \widehat{\mathfrak{sl}}_n$), the proof of Theorem D uses an entirely different method. The geometric descriptions of the shuffle algebra and \mathcal{P} are essential in the establishment of this injective map. Another ingredient is the verification of the quadratic relation and the Serre relation in the shuffle algebra, which is established using a symmetric polynomial identity proved in Appendix A, which might be interesting on its own right.

The shuffle description of $U_q^+(\widehat{\mathfrak{g}})$ was used to provide a combinatorial characterization of the dual canonical basis, and hence the shuffle formula became a basic calculation tool in q -characters of the KLR algebra [Lec04, KR11]. By analogy, we expect the shuffle description of the Yangian to lead to an analogue of the dual canonical basis in the Yangian. Furthermore, according to the recent progresses [BHLW14, SVV14], it is expected that the trace of the graded representation category of KLR algebra is isomorphic to the Yangian. Thus, the shuffle description of Yangian should provide a calculation tool in the study of trace decategorification of KLR algebra.

0.5. K -theory and elliptic cohomology. In the present paper, for simplicity we assume the coefficient ring of A is a \mathbb{Q} -algebra. Over a \mathbb{Q} -algebra, all the formal group laws are isomorphic. As a consequence, all the OCT considered here are related to the Chow group by Riemann-Roch-type theorems. Therefore, the convolution algebras using any OCT will be isomorphic to the convolution algebra using Chow group. However, the algebra ${}^A\mathcal{P}(Q)$ depends on the formal group law itself, instead of the isomorphism class of the formal group law (see [YZ1]). Consequently, ${}^A\mathcal{P}(Q)$ is not isomorphic to ${}^{\text{CH}}\mathcal{P}(Q)$. Nevertheless, motivated by the isomorphism between the completion of the Yangian and the quantum loop algebra, constructed by Gautam-Toledano Laredo in [GTL10], it is reasonable to expect the difference to be resolved by adding the Cartan part, i.e., certain completion of the extended spherical subalgebra ${}^A\mathcal{P}(Q)^{s,\epsilon}$ is isomorphic to a completion of ${}^{\text{CH}}\mathcal{P}(Q)^{s,\epsilon}$.

The study of various quantum groups and their representations in the affine setting has been a fruitful subject. Amongst the interesting results, a construction of an equivalence between the finite dimensional representations of $U_q(L\mathfrak{g}_Q)$ and $Y_h(\mathfrak{g}_Q)$ was given by Gautam and Toledano Laredo [GTL10, GTL13, GTL14].

The present paper contributes more members to the family of quantum groups in the affine setting. When A is the K -theory, it is expected in [Gr94b] that, after suitable modification, there is an algebra isomorphism $U_q^+(L\mathfrak{g}) \rightarrow {}^K\mathcal{P}^s(Q)$. However, the relation between the ${}^K\widetilde{\mathcal{P}}^s(Q)$ action on equivariant K -theory of quiver varieties and the action of $U_q(L\mathfrak{g})$ studied in [Nak01] is not clear. As the study of this relation would involve a different set of generators of $U_q^+(L\mathfrak{g})$ than the one used in [Nak01], we do not achieve this in the current paper.

In a future publication [YZ2], we will study the preprojective CoHA when A is equivariant elliptic cohomology. The extended spherical subalgebra is a bialgebra, whose Drinfeld double is the elliptic quantum group defined by Felder and collaborators (see e.g., [Fed94]). The algebraic approach to study the representations of the elliptic quantum group is carried out in the work of [GTL15].

The idea of the geometric approach of studying elliptic quantum groups using elliptic cohomology goes back to [Gr94a] and [GKV95]. In [GKV95], an axiomatic definition of the equivariant elliptic cohomology was given. It is shown in [GKV95] that the classical elliptic algebra of \mathfrak{gl}_n acts on the equivariant elliptic cohomology of the variety of n -step flags. In [ZZ15], it is proved that the elliptic affine Hecke algebra acts on the equivariant elliptic cohomology of the Springer fibers. We postpone to future publications the relation between the representations in [ZZ15] and the representations of the elliptic quantum groups.

Parallel to this paper, we would like to mention that in [ZZ14], it is proved that the formal affine Hecke algebra studied in [HMSZ12] acts on the A -homology of the Springer fibers in cotangent bundle of flag varieties. We expect that when $\mathfrak{g}_Q = \mathfrak{sl}_n$ the representations of the formal affine Hecke algebra in [ZZ14] and the representation of \mathcal{P} studied in the current paper are related by Schur-Weyl duality. However, we postpone the investigation to later publications.

0.6. Preprojective CoHA and critical CoHA. Associated to the 3-Calabi-Yau category of representations of a quiver with potential, there is a critical CoHA defined by Kontsevich-Soibelman, in the framework of Donaldson-Thomas theory. The preprojective CoHA is associated to the 2-Calabi-Yau category of representations of the preprojective algebra of the quiver Q . For the class of quivers with potential studied by Ginzburg in [Gin09], we construct an isomorphism of the corresponding preprojective CoHA and the critical CoHA in [YZ16].

More precisely, for an arbitrary quiver Q , let (\hat{Q}, W) be the extended quiver with potential defined by Ginzburg in [Gin09, (4.2.3)]. Let $\mathcal{H} = \bigoplus_{v \in \mathbb{N}^I} \mathcal{H}_v$ be the critical CoHA associated to (\hat{Q}, W) defined by Kontsevich and Soibelman [KoSo11] with a suitable extra 2-dimensional torus action (see [YZ16] for the details). Let ${}^{\text{BM}}\mathcal{P} = \bigoplus_{v \in \mathbb{N}^I} H_{G_v \times T}^{\text{BM}}(\mu_v^{-1}(0); \mathbb{Q})$ be the preprojective CoHA in the topological setting of Borel-Moore homology. We show in [YZ16] is an isomorphism of \mathbb{N}^I -graded associative algebras $\Xi : {}^{\text{BM}}\mathcal{P} \rightarrow \mathcal{H}$.

A direct consequence of Theorems D and [YZ16] is the existence of an injective algebra homomorphism $Y_h^+(\mathfrak{g}_Q) \rightarrow \widetilde{\mathcal{H}}|_{t_1=t_2=\hbar/2}$, where $\widetilde{\mathcal{H}}$ is the sign twist of \mathcal{H} coming from the Euler-Ringel form.

After an earlier version of this paper appeared on arXiv, in the course of studying the semi-canonical bases and an analogue of the Kac polynomial, a similar notion of CoHA for preprojective algebra was introduced in [RS15], although the results in the present paper were not included in *loc. cit.* except for Theorem A in the case when A is the Borel-Moore homology.

Shortly after an earlier version of the current paper, which contains Theorems D for the Dynkin case and [YZ16, Theorem A], appeared on arXiv, a conjecture about a relation between Yangian and critical CoHA for more general cases was independently proposed by Davison [Da15]. Inspired by Davison's conjecture, the authors, based on the same method as in the Dynkin case, proved Theorems D in the current generality. It turns out that [MO12, Theorem 5.5.1] significantly simplifies our argument for \mathfrak{g}_Q outside of the Dynkin case.

Summary. For the convenience of the readers, we summarize the various algebras as follows.

$$\begin{array}{ccccccc}
 & \text{CH}\mathcal{P}^{\mathfrak{s},\epsilon} & \xrightarrow{\quad} & \text{CH}\mathcal{P}^{\mathfrak{s},\epsilon} & & & \\
 & \downarrow & & \downarrow \cong & & & \\
 \text{CH}\mathcal{P} & \subset \text{CH}\mathcal{P}^{\epsilon} & \xrightarrow{\quad} & \text{CH}\mathcal{P}^{\epsilon} & \xrightarrow{\quad} & \text{End } \mathcal{M}(w) & \\
 \downarrow & \downarrow & & \downarrow & & \uparrow & \\
 \mathcal{SH} & \subset \mathcal{SH}^{\epsilon} & \longrightarrow & \mathcal{SH}^{\epsilon}|_{t_1=t_2=\frac{\hbar}{2}} & \longleftrightarrow & Y_h^{\geq 0} & \subset Y_{\text{MO}}
 \end{array}$$

Acknowledgments. Many ideas in this paper are rooted in the work of Schiffmann and Vasserot [SV10, SV12]. The authors are grateful to Marc Levine, Zongzhu Lin, Yan Soibelman, Valerio Toledano Laredo, and Eric Vasserot for helpful discussions and correspondence. Section §7.3 is motivated by a discussion with Ben Davison at Park City Mathematics Institute in 2015.

Part of this paper was conceived when G.Z. was waiting for the security clearance of his US-Visa in Beijing in 2013. He would therefore like to thank the Morningside Center of Mathematics at the Chinese Academy of Sciences for accommodation, and the U.S. overseas diplomats for the otherwise unavailable opportunity of this stimulating visit. During the preparation of this paper, G.Z. was hosted by Max-Planck-Institut für Mathematik in Bonn. During the revision of this paper, Y.Y. was hosted by MPIM and G.Z. was supported by Fondation Sciences Mathématiques de Paris and CNRS.

1. ALGEBRAIC ORIENTED COHOMOLOGY THEORY

In this section we collect basic notions about equivariant oriented Borel-Moore homology theory.

1.1. Oriented Borel-Moore homology theory. Fix a base field k . We denote by \mathbf{Sch}_k the category of separated schemes of finite type over k and by \mathbf{Sm}_k its full subcategory consisting of smooth quasi-projective k -schemes. Let R be a commutative ring, and let (R, F) be a formal group law over R , that is, $F(x, y) \in R[[x, y]]$ such that

$$F(x, y) = F(y, x), \quad F(x, 0) = x, \quad F(x, F(y, z)) = F(F(x, y), z).$$

For simplicity, we assume R is a \mathbb{Q} -algebra.

Example 1.1. For any commutative graded ring R , the element $F_a(x, y) = x + y$ in $R[[x, y]]$ defines the *additive formal group law*.

For any commutative graded ring R , the element $F_m(x, y) = x + y - \beta xy$ in $R[[x, y]]$ with $\beta \in R$ defines a *multiplicative formal group law*.

There is a *universal formal group law* $(\mathbb{L}az, F_{\mathbb{L}az})$, whose coefficient ring $\mathbb{L}az$, called the Lazard ring, is a polynomial ring in countably many generators over \mathbb{Z} . For any formal group law (R, F) , there exists a unique ring homomorphism $\phi_F : \mathbb{L}az \rightarrow R$ such that $F = \phi_F(F_{\mathbb{L}az})$.

Let Ω be the algebraic cobordism of Levine-Morel defined in [LM07]. Then $A = \Omega \otimes_{\mathbb{L}az} R$ defines an oriented Borel-Moore homology theory in the sense of [LM07, Definition 2.2.1, Theorem 7.11]. More precisely, it is the following data:

- (1) For any object X in \mathbf{Sch}_k , $A(X)$ is a graded abelian group.
- (2) (Proper pushforward) For $f : Y \rightarrow X$ a proper morphism in \mathbf{Sch}_k , there is a homomorphism $f_* : A(Y) \rightarrow A(X)$.
- (3) (Smooth pullback) For a smooth morphism $f : Y \rightarrow X$ in \mathbf{Sch}_k , there is a graded homomorphism $f^* : A(X) \rightarrow A(Y)$.
- (4) (Gysin pullback) For any local complete intersection morphism $f : Y \rightarrow X$, and an arbitrary morphism $Z \rightarrow X$, let $g : Z \times_X Y \rightarrow Z$ be the base change. Then, there is a refined pullback map $f_g^\sharp : A(Z) \rightarrow A(Z \times_X Y)$. We will also write f^\sharp if g is understood from the context. It specializes to the smooth pullback f^* when f is smooth and $Z \rightarrow X$ is the identity morphism on X .
- (5) (1st Chern class operators) For each line bundle $L \rightarrow X$, $X \in \mathbf{Sch}_k$, there is a graded homomorphism $\tilde{c}_1(L) : A(X) \rightarrow A(X)$. For line bundles L, M on X , the operators $\tilde{c}_1(L)$ and $\tilde{c}_1(M)$ commute.

- (6) (External products) For $X, Y \in \mathbf{Sch}_k$, there is a graded homomorphism $A(X) \otimes_{\mathbb{Z}} A(Y) \rightarrow A(X \times_k Y)$, which is commutative and associative in the obvious sense. There is an element $1 \in A(\text{pt})$ which, together with the external product $A(\text{pt}) \otimes_{\mathbb{Z}} A(\text{pt}) \rightarrow A(\text{pt})$, makes $A(\text{pt})$ into a commutative graded ring with unit.
- (7) (Formal group law) There is a homomorphism $\phi_A : \mathbb{L}az \rightarrow A(\text{pt})$ of graded rings, such that, letting $F_A(u, v) \in A(\text{pt})[[u, v]]$ be the image of the universal formal group law with respect to ϕ_A , for each $X \in \mathbf{Sch}_k$ and each pair of line bundles L, M on X , we have

$$\tilde{c}_1(L \otimes M) = F_A(\tilde{c}_1(L), \tilde{c}_1(M)).$$

These all satisfy a number of compatibilities, detailed in [LM07, § 2.1, § 2.2]. As we do not consider homological degree, we will simply treat $A(X)$ as an abelian group by forgetting the grading. As we will need the compatibility of push-forward and the refined Gysin pull-backs, we collect some basic facts in § 1.4.

When restricting A to the category \mathbf{Sm}_k , it factors through the category of commutative graded rings with unit. Indeed, when X is smooth, the diagonal embedding $\Delta : X \rightarrow X \times X$ is a local complete intersection morphism. The ring structure on $A(X)$ is obtained from $\Delta^* \circ \boxtimes : A(X) \otimes A(X) \rightarrow A(X \times X) \rightarrow A(X)$. In this case, for any line bundle L , the first Chern class $c_1(L) = \tilde{c}_1(L)(1_X) \in A(X)$ is well-defined.

1.2. Equivariant oriented cohomology theories. We will consider equivariant Borel-Moore homology theory in the sense of [CZZ14, § 2] and [ZZ14, § 5.1]. Examples includes the equivariant K -theory of [T93], the equivariant Chow ring (the direct sum of the graded pieces defined in [EG98]), and the following construction.

For any reductive algebraic group G , let \mathbf{Sch}_k^G be the category of schemes over k of finite type with a G -action. Following Totaro's construction [T99], in [HM13] it has been explained how A extends to an equivariant Borel-Moore homology theory. More precisely, for any reductive group G , the classifying space of G is a system $EG := \{EG_N\}_{N \in \mathbb{N}}$, where each EG_N is a Zariski open subset in a representation of G on which G -acts freely, and satisfies the condition of a *good system* spelled out in [HM13, Definition 10]. For simplicity, we call $BG := \{EG_N/G\}_{N \in \mathbb{N}}$ the *classifying space* of G , and we denote $\lim_N A(X \times_G EG_N)$ by $A_G(X)$.

The functor sending any $X \in \mathbf{Sch}_k^G$ to $A_G(X)$ is endowed with similar structures as the ordinary oriented Borel-Moore homology A , e.g., equivariant proper pushforward, equivariant Gysin pullback, equivariant Chern classes in A_G , which satisfy the usual compatibility conditions. When G is trivial, A_G is an oriented Borel-Moore homology theory in the sense of § 1.1.

Examples of equivariant oriented Borel-Moore homology theories are given by the equivariant intersection theory CH_G by Edidin and Graham [EG98], and the equivariant algebraic cobordism Ω_G in [Des09], [HM13], [Kr12].

Now we study in details the infinite Grassmannian which is the classifying space of GL_r . The infinite Grassmannian $\text{Grass}(r, \infty)$ is defined as the system

$$\text{Grass}(r, \infty) := \{\text{Grass}(r, N)\}_{N \in \mathbb{N}}.$$

Let $\mathcal{R}(r) := \{\mathcal{R}(r, N)\}_{N \in \mathbb{N}}$ with $\mathcal{R}(r, N)$ being the tautological rank r vector bundle on $\text{Grass}(r, N)$. Let $E\text{GL}_r := \{E\text{GL}(r, N)\}_{N \in \mathbb{N}}$ with $E\text{GL}(r, N)$ being the frame bundle of $\mathcal{R}(r, N)$, that is, the scheme representing the GL_r -torsor $\underline{\text{Isom}}(\mathcal{O}^r, \mathcal{R}(r, N))$. In particular, $E\text{GL}(r, N)$ is a Zariski open subset in a GL_r -representation, and the system $E\text{GL}_r = \{E\text{GL}(r, N)\}_N$ satisfies the conditions of a good system.

Assume $r = r_1 + r_2$, where r_1 and r_2 are two positive integers. Let L be the Levi-subgroup $\text{GL}_{r_1} \times \text{GL}_{r_2} \subset \text{GL}_r$. The classifying space BL is $\text{Grass}(r_1, \infty) \times \text{Grass}(r_2, \infty)$. We have the

natural morphism

$$\mathrm{Grass}(r_1, \infty) \times \mathrm{Grass}(r_2, \infty) \rightarrow \mathrm{Grass}(r, \infty)$$

sending a pair of subspaces $(\mathbb{A}^{r_1} \subset \mathbb{A}^{N_1}, \mathbb{A}^{r_2} \subset \mathbb{A}^{N_2})$ to $(\mathbb{A}^{r_1} \oplus \mathbb{A}^{r_2}) \subset \mathbb{A}^{N_1+N_2}$, for N_1, N_2 large enough. It identifies $\mathrm{Grass}(r_1, \infty) \times \mathrm{Grass}(r_2, \infty)$ with the total space of the Grassmannian bundle $\mathrm{Grass}(r_1, \mathcal{R}(r))$ on $\mathrm{Grass}(r, \infty)$. In other words, let P be the parabolic subgroup consisting of block-upper triangular matrices containing L as its Levi-subgroup. We will identify EL with EG . Then, the classifying space BL is realized as EG/P . Summarize the notations in the following diagram.

$$(1) \quad \begin{array}{ccc} \mathrm{Grass}(r_1, \infty) \times \mathrm{Grass}(r_2, \infty) & \xrightarrow{\cong} & \mathrm{Grass}(r_1, \mathcal{R}(r)) \\ \downarrow & \swarrow p & \\ \mathrm{Grass}(r, \infty) & & \end{array}$$

Let \mathfrak{S}_n be the symmetric group on n elements. By the usual calculation (see, e.g., [PPR08, Theorem 2.2]) we have the following.

Lemma 1.2. *Let $\lambda_1, \dots, \lambda_r$ be the Chern roots of $\mathcal{R}(r)$ on $\mathrm{Grass}(r, \infty)$. For any oriented cohomology theory A ,*

$$A(\mathrm{Grass}(r, \infty)) = A(pt)[[\lambda_1, \dots, \lambda_r]]^{\mathfrak{S}_r}.$$

As a consequence, we get:

Lemma 1.3. *With Notations as above, let $\lambda_1, \dots, \lambda_r$ be the Chern roots of the tautological rank r bundle $\mathcal{R}(r)$ on $\mathrm{Grass}(r, \infty)$, then we have*

$$A(\mathrm{Grass}(r_1, \mathcal{R}(r))) = A(\mathrm{Grass}(r_1, \infty) \times \mathrm{Grass}(r_2, \infty)) \cong A(pt)[[\lambda_1, \dots, \lambda_r]]^{\mathfrak{S}_{r_1} \times \mathfrak{S}_{r_2}}.$$

1.3. Quillen-Weyl character formula. Let $V \rightarrow X$ be a vector bundle of X with Chern roots $\lambda_1, \dots, \lambda_n$. For $0 < r \leq n$, denote the corresponding Grassmannian bundle by $\mathrm{Grass}(r, V)$ with $p : \mathrm{Grass}(r, V) \rightarrow X$ the bundle map. In particular, when $r = 1$, we get the projective bundle $\mathrm{Grass}(1, V) = \mathbb{P}(V)$.

It is known that the cohomology of Grassmannian is generated by the Chern classes of the tautological vector bundle. In other words, let v_1, \dots, v_r be the Chern roots of the rank- r tautological bundle $\mathcal{R}(r)$ on $\mathrm{Grass}(r, V)$. We have the following.

Lemma 1.4. *For any oriented cohomology theory A , extended to an equivariant Borel-Moore homology by Totaro's construction, there is an algebra epimorphism*

$$A(X)[[v_1, \dots, v_r]]^{\mathfrak{S}_r} \twoheadrightarrow A(\mathrm{Grass}(r, V)).$$

In the rest of this subsection, we give a pushforward formula of the Grassmannian bundle in arbitrary oriented cohomology theory. We start with the pushforward formula in the intersection theory CH.

For any pair (p, q) of positive integers, let $\mathrm{Sh}(p, q)$ be the subset of \mathfrak{S}_{p+q} consisting of (p, q) -shuffles (permutations of $\{1, \dots, n\}$ that preserve the relative order of $\{1, \dots, p\}$ and $\{p+1, \dots, p+q\}$).

Proposition 1.5. *Let $X \in \mathbf{Sm}_k$ be a smooth quasi-projective variety, V be some n -dimensional vector bundle on X , and $p : \mathrm{Grass}(r, V) \rightarrow X$ be the corresponding Grassmannian bundle. For any $f(v_1, \dots, v_r) \in \mathrm{CH}^*(\mathrm{Grass}(r, V))$ as in Lemma 1.4, we then have*

$$p_*(f(v_1, \dots, v_r)) = \sum_{\sigma \in \mathrm{Sh}(r, n-r)} \sigma \frac{f(\lambda_1, \dots, \lambda_r)}{\prod_{1 \leq j \leq r, r+1 \leq i \leq n} (\lambda_i - \lambda_j)},$$

where $\lambda_1, \dots, \lambda_n$ are the Chern roots of V .

Proof. This result is well-known. Nevertheless, for the convenience of the readers, we sketch the proof here. Using the argument in [PS04], without loss of generality we assume the vector bundle V splits into line bundles $\bigoplus_{i=1}^n L_i$. Let the torus $T := (\mathbb{G}_m)^n$ act on V fiberwise. The i -th copy of \mathbb{G}_m acting on L_i by dilation. It induces an action of T on $\text{Grass}(r, V)$. The fixed point set is

$$\{W_J := \bigoplus_{j \in J} L_j \subset V \mid J \subset [1, \dots, n], |J| = r\}.$$

At the fixed point W_J , the tangent space is $T_{W_J} \text{Grass}(r, V) = \bigoplus_{j \in J, i \notin J} L_j^* \otimes L_i$. Therefore, the Euler class of $T_{W_J} \text{Grass}(r, V)$ is given by $e(T_{W_J} \text{Grass}(r, V)) = \prod_{j \in J, i \notin J} (\lambda_i - \lambda_j)$. Now by the Atiyah-Bott localization formula, proved in the setting of intersection theory in [EG98], we have

$$p_*(f(v_1, \dots, v_r)) = \sum_{\{\sigma \in \text{Sh}(r, n-r)\}} \sigma \frac{f(\lambda_1, \dots, \lambda_r)}{\prod_{1 \leq j \leq r, r+1 \leq i \leq n} (\lambda_i - \lambda_j)}.$$

□

We use the short-hand notations $u +_F v := F(u, v)$. Denote $-_F v$ to be the inverse of v of the formal group law, in other words, $F(v, -_F v) = 0$.

Proposition 1.6. *Let $V \rightarrow X$ be a rank n vector bundle and let $p : \text{Grass}(r, V) \rightarrow X$ be the associated Grassmannian bundle. For any the oriented cohomology theory A , extended to an equivariant Borel-Moore homology by Totaro's construction, with formal group law (R, F) , let $f(v_1, \dots, v_r) \in A(\text{Grass}(r, V))$. Then,*

$$p_*^A(f(v_1, \dots, v_r)) = \sum_{\{\sigma \in \text{Sh}(r, n-r)\}} \sigma \frac{f(\lambda_1, \dots, \lambda_r)}{\prod_{1 \leq j \leq r, r+1 \leq i \leq n} (\lambda_i -_F \lambda_j)},$$

where $\lambda_1 \dots \lambda_n$ are Chern roots of V in A .

Proof. Let t_1, \dots, t_n be Chern roots of V in CH^* . Denote the exponential of the formal group law F by λ_τ . That is, a formal series in $R[[u]]$, such that $F(\lambda_\tau(u), \lambda_\tau(v)) = \lambda_\tau(u+v)$. Then we have, the Chern roots of V in A are $\{\lambda_i := \lambda_\tau(t_i)\}_{i=1, \dots, n}$. We denote $\lambda = (\lambda_1, \dots, \lambda_n)$, and $t = (t_1, \dots, t_n)$ for short. By [LYZ13, Proposition 1.13(3)], we have

$$p_*^A(f(v)) = p_*^{\text{CH}}(\text{Td}(T_p) f(\lambda_\tau(t))) = p_*^{\text{CH}}\left(\frac{\prod_{j \in J, i \notin J} (t_i - t_j)}{\prod_{j \in J, i \notin J} (\lambda_\tau(t_i - t_j))} f(\lambda_\tau(t))\right).$$

Now we can use the pushforward formula in Proposition 1.5. Thus,

$$p_*^A(f(v)) = \sum_{\{\sigma \in \text{Sh}(r, n-r)\}} \sigma \frac{f(\lambda_1, \dots, \lambda_n)}{\prod_{1 \leq j \leq r, r+1 \leq i \leq n} (\lambda_i -_F \lambda_j)}.$$

□

The formula in Proposition 1.6 will be referred to as the Quillen-Weyl formula. When A is the equivariant K -theory, the formula specializes to the familiar Weyl character formula. In the special case in Proposition 1.8 bellow, it is due to Quillen.

We apply the pushforward formula to the situation, when $X = \text{Grass}(n, \infty)$ and $V = \mathcal{R}(n)$ the tautological rank- n vector bundle. For $0 < r \leq n$, recall the classifying space $B\text{GL}_r \times B\text{GL}_{n-r}$ was identified with $\text{Grass}(r, \mathcal{R}(n))$ in (1).

Corollary 1.7. *Let $p : B \mathrm{GL}_r \times B \mathrm{GL}_{n-r} \cong \mathrm{Grass}(r, \mathcal{R}(n)) \rightarrow \mathrm{Grass}(n, \infty) \cong B \mathrm{GL}_n$ be the natural projection. Then the push-forward map*

$$p_* : A(B \mathrm{GL}_r \times B \mathrm{GL}_{n-r}) \cong A(pt)[[\lambda_1, \dots, \lambda_n]]^{\mathfrak{S}_r \times \mathfrak{S}_{n-r}} \rightarrow A(B \mathrm{GL}_n) \cong A(pt)[[\lambda_1, \dots, \lambda_n]]^{\mathfrak{S}_n}$$

is given by

$$f(\lambda_1, \dots, \lambda_n) \mapsto \sum_{\{\sigma \in \mathrm{Sh}(r, n-r)\}} \sigma \frac{f(\lambda_1, \dots, \lambda_n)}{\prod_{1 \leq j \leq r, r+1 \leq i \leq n} (\lambda_i -_F \lambda_j)},$$

where $\lambda_1 \dots \lambda_n$ are Chern roots of $\mathcal{R}(n)$ in A .

In particular, when $r = 1$, $\mathrm{Grass}(r, V) \cong \mathbb{P}(V)$, Proposition 1.6 yields the following.

Proposition 1.8 (See also [Vi07]). *Let (R, F) be a formal group law, and R^* be the corresponding cohomology theory. Let $X \in \mathbf{Sm}_k$ be a smooth quasi-projective variety, V be some n -dimensional vector bundle on X , and $\pi : \mathbb{P}_X(V) \rightarrow X$ be the corresponding projective bundle. We write $t = c_1(\mathcal{O}(1))$. Then for any $f(t) \in R^*(X)_{\mathbb{Q}}[[t]]$, we have*

$$\pi_*^R(f(c_1(\mathcal{O}(1)))) = \sum_i \frac{f(-_F \lambda_i)}{\prod_{j \neq i} (\lambda_j -_F \lambda_i)},$$

where π_*^R is the push-forward in the theory $R_{\mathbb{Q}}^*$, $\lambda_1, \dots, \lambda_n$ are the Chern roots of V in $R_{\mathbb{Q}}^*$.

Let I be a finite set and $v = (v^i)_{i \in I} \in \mathbb{N}^I$ be a vector with entries non-negative integers. For $G_v = \prod_{i \in I} \mathrm{GL}_{v^i}$, we have $BG_v \cong \mathrm{Grass}(v, \infty) := \prod_{i \in I} \mathrm{Grass}(v^i, \infty)$. For each $i \in I$, denote the Chern roots of $\mathcal{R}(v^i)$ on the factor $\mathrm{Grass}(v^i, \infty)$ by λ_j^i with $j = 1, \dots, v^i$. Then the group $\mathfrak{S}_v := \prod_{i \in I} \mathfrak{S}_{v^i}$ acts on $A(pt)[[\lambda_j^i]_{i \in I, j=1, \dots, v^i}]$ by permuting the Chern roots. In this setup, we have

$$A(\mathrm{Grass}(v, \infty)) \cong A(pt)[[\lambda_j^i]_{i \in I, j=1, \dots, v^i}]^{\mathfrak{S}_v}.$$

For any dimension vector $v \in \mathbb{N}^I$, with $v = v_1 + v_2$, we denote $\mathrm{Sh}(v_1, v_2) \subset \mathfrak{S}_v$ to be the product $\prod_{i \in I} \mathrm{Sh}(v_1^i, v_2^i)$.

Remark 1.9. When A is the K -theory (resp. the intersection theory) with \mathbb{Q} -coefficients, we will use the equivariant K -theory of [T93] (resp. the direct sum of the graded pieces defined in [EG98]). Hence, $K_{\mathbb{G}_m}(pt) = \mathbb{Q}[z^{\pm}]$ (resp. $\mathrm{CH}_{\mathbb{G}_m}(pt) = \mathbb{Q}[z]$). The formula in Proposition 1.6 still holds.

1.4. Lagrangian correspondence formalism. Now we recall the Lagrangian correspondence formalism following the exposition in [SV12].

Let X be a smooth quasi-projective variety endowed with an action of a reductive algebraic group G . The cotangent bundle T^*X is a symplectic variety. The induced action of G on T^*X is Hamiltonian. Let $\mu : T^*X \rightarrow (\mathrm{Lie} G)^*$ be the moment map. Following [SV12], we denote $\mu^{-1}(0) \subseteq T^*X$ by T_G^*X .

Let $P \subset G$ be a parabolic subgroup and $L \subset P$ be a Levi subgroup. Let Y be a smooth quasi-projective variety equipped an action of L , and X' smooth quasi-projective with a G -action. Let $\mathcal{V} \subseteq Y \times X'$ be a smooth subvariety. Let $\mathrm{pr}_1, \mathrm{pr}_2$ be the two projections restricted on \mathcal{V}

$$Y \xleftarrow{\mathrm{pr}_1} \mathcal{V} \xrightarrow{\mathrm{pr}_2} X'.$$

Assume the first projection pr_1 is a vector bundle, and the second projection pr_2 is a closed embedding.

Let $X := G \times_P Y$ be the twisted product. Set $W := G \times_P \mathcal{V}$ and consider the following maps

$$X \xleftarrow{f} W \xrightarrow{g} X'$$

$$f : [(g, v)] \mapsto [(g, \text{pr}_1(v))], \quad g : [(g, v)] \mapsto g \text{pr}_2(v),$$

where $[(g, v)]$ is the pair $(g, v) \bmod P$. Note that the natural map $T^*X \rightarrow G \times_P T^*Y$ is a vector bundle. The following lemma is proved in [SV12].

Lemma 1.10 ([SV12]). *There is an isomorphism $G \times_P T_L^*Y \cong T_G^*X$ such that the following diagram commutes*

$$\begin{array}{ccc} G \times_P T_L^*Y & \xrightarrow{\cong} & T_G^*X \\ \downarrow & & \downarrow \\ G \times_P T^*Y & \hookrightarrow & T^*X \end{array}$$

where $G \times_P T^*Y \hookrightarrow T^*X$ is the zero-section of the vector bundle $T^*X \rightarrow G \times_P T^*Y$.

Let $Z := T_W^*(X \times X')$ be the conormal bundle of W in $X \times X'$. Let $Z_G \subseteq T_G^*X \times T_G^*X'$ be the intersection $Z \cap (T_G^*X \times T_G^*X')$. Then we have the following diagram.

$$\begin{array}{ccccccc} G \times_P T_L^*Y & \xrightarrow{\cong} & T_G^*X & \xleftarrow{\bar{\phi}} & Z_G & \xrightarrow{\bar{\psi}} & T_G^*X' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G \times_P T^*Y & \hookrightarrow & T^*X & \xleftarrow{\phi} & Z & \xrightarrow{\psi} & T^*X' \end{array}$$

where $\phi : Z \rightarrow T^*X$ and $\psi : Z \rightarrow T^*X'$ are respectively the first and second projections of $T^*X \times T^*X'$ restricted to Z .

Lemma 1.11. [SV12, Lemma 7.3] *The morphism $\psi : Z \rightarrow T^*X'$ is proper. We have $\psi^{-1}(T_G^*X') = Z_G$ and $\phi^{-1}(T_G^*X) = Z_G$.*

Let A be an oriented Borel-Moore homology theory. Existence of refined Gysin pull-back and Lemma 1.11 ensure the existence of the map $\bar{\psi}_* \circ \phi^\sharp : A_G(T_G^*X) \rightarrow A_G(T_G^*X')$.

Lemma 1.12. *The following diagram commutes*

$$\begin{array}{ccc} A_G(T_G^*X) & \xrightarrow{\bar{\psi}_* \circ \phi^\sharp} & A_G(T_G^*X') \\ \downarrow & & \downarrow \\ A_G(T^*X) & \xrightarrow{\psi_* \circ \phi^*} & A_G(T^*X') \end{array}$$

where the vertical maps are push-forwards induced by natural embeddings.

Proof. It follows directly from Lemma 1.13(1) below. □

1.5. Base-change for Lagrangian correspondences. We collect some basic facts about compatibility of push-forward and Gysin pull-back in oriented Borel-Moore homology. We will apply these facts to the setting of Lagrangian correspondences.

Recall that two morphisms $f : Y \rightarrow X$ and $q : X' \rightarrow X$ are said to be *transversal* if

$$\text{Tor}_k^{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y) = 0, \quad \text{for any } k > 0.$$

Lemma 1.13 ([LM07], Theorem 6.6.6(2), and Lemma 6.6.2). *Consider the following diagram in Sch_k*

$$\begin{array}{ccccc} H & \xrightarrow{g'} & Y' & \longrightarrow & Y \\ \downarrow f'' & & \downarrow f' & & \downarrow f \\ W & \xrightarrow{g} & X' & \xrightarrow{q} & X \end{array}$$

with all squares Cartesian. Assume f is a locally complete intersection morphism.

- (1) If g is proper, then $f_{f',g*}^\# = g'_* f_{f''}^\#$.
(2) If f and g are transversal, then $f_{f''}^\# = f_{f'''}^\#$.

As a consequence, we have the following.

Lemma 1.14. *Consider the following commutative diagram in which every square is Cartesian.*

$$\begin{array}{ccccc}
W' & \xrightarrow{f''} & Y' & & \\
\downarrow g'' & \searrow & \downarrow g' & \searrow & \\
Z' & \xrightarrow{f'} & W & \xrightarrow{\quad} & Y \\
\downarrow l & \searrow & \downarrow i & \searrow & \downarrow g \\
Z & \xrightarrow{f} & X' & \xrightarrow{i} & X
\end{array}$$

Assume g and i are proper, f and g are transversal, and f is a locally complete intersection morphism. Then we have the equality

$$f_* \circ l_{g''}^\# = g_{g'}^\# \circ f'_*$$

as homomorphisms from $A(Z')$ to $A(Y')$.

Proof. By Lemma 1.13(1), we have $f_* \circ g_{g''}^\# = g_{g'}^\# \circ f'_*$. By Lemma 1.13(2), $g_{g''}^\# = l_{g''}^\#$. We are done. \square

The following is a sufficient condition for two morphisms to be transversal.

Lemma 1.15 ([SV12], Proposition C.1). *Consider the following Cartesian diagram*

$$\begin{array}{ccc}
Y' & \xrightarrow{f'} & X' \\
\downarrow g' & & \downarrow g \\
Y & \xrightarrow{f} & X
\end{array}$$

Assume g is proper, the map $f' \times g' : Y' \rightarrow X' \times Y$ is a closed embedding, and assume $\dim(X) + \dim(Y') = \dim(X') + \dim(Y)$. Then, g' is proper, and f and g are transversal.

One example is given as below. Let $W_1 \subset X_3 \times X_2$, $W_2 \subset X_3 \times X_1$, and $W_3 \subset X_2 \times X_1$ be subvarieties. We assume $W_2 = W_1 \times_{X_2} W_3$. We consider the Lagrangian subvarieties

$$Z_1 = T_{W_1}^*(X_3 \times X_2), \quad Z_2 = T_{W_2}^*(X_3 \times X_1), \quad Z_3 = T_{W_3}^*(X_2 \times X_1).$$

Assume the intersection $(W_1 \times X_1) \cap (X_3 \times W_2)$ is transversal in $X_3 \times X_2 \times X_1$. Thus, by [CG, Theorem 2.7.26] we have an isomorphism $Z_1 \times_{T^*X_2} Z_3 \cong Z_2$. In particular, the following commutative diagram is Cartesian

$$\begin{array}{ccc}
Z_1 & \xrightarrow{\phi_1} & T^*X_2 \\
\uparrow & & \uparrow \psi_3 \\
Z_2 & \longrightarrow & Z_3
\end{array}$$

The following lemma is a direct consequence of Lemma 1.15.

Lemma 1.16. *With notations as above, we have $\dim(Z_1) + \dim(Z_3) = \dim(Z_2) + \dim(T^*X_2)$. In particular, ϕ_1 and ψ_3 are transversal. The map $Z_2 \rightarrow Z_1$ is proper.*

2. THE FORMAL COHOMOLOGICAL HALL ALGEBRAS

In this section, we review in the algebraic setting the cohomological Hall algebra defined by Kontsevich and Soibelman in [KoSo11]. The idea of studying CoHA from arbitrary oriented Borel-Moore homology theory goes back to [KoSo11, § 3.7]. We spell out in an explicit fashion the shuffle formula for this CoHA. We emphasize the dependence of the shuffle formula on the formal group law.

2.1. The formal cohomological Hall algebras. Let k be an arbitrary field. Let Q be a quiver with vertex set I and arrow set H . We assume in this paper that I and H are finite sets. For $h \in H$, we denote by $\text{in}(h)$ (resp. $\text{out}(h)$) the incoming (resp. outgoing) vertex of h . For any dimension vector $v = (v^i)_{i \in I} \in \mathbb{N}^I$, the representation space of Q with dimension vector v is denoted by $\text{Rep}(Q, v)$. That is, let $V = \{V^i\}_{i \in I}$ be an I -tuple of k -vector spaces with dimension vector $\dim(V^i) = v^i$. Then,

$$\text{Rep}(Q, v) := \bigoplus_{h \in H} \text{Hom}_k(V^{\text{out}(h)}, V^{\text{in}(h)}).$$

The algebraic group $G_v := \prod_{i \in I} \text{GL}(v^i)$ acts on $\text{Rep}(Q, v)$ by conjugation. For any $v_1, v_2 \in \mathbb{N}^I$ such that $v := v_1 + v_2$, let $V_1 \subset V$ be an I -tuple of vector subspaces of V with dimension vector v_1 . The parabolic subgroup of G_v that preserves V_1 will be denoted by $P \subset G_v$. Let $L = G_{v_1} \times G_{v_2}$ be the standard Levi-subgroup in P .

Let A be an oriented Borel-Moore homology theory as in § 1 and (R, F) be the formal group law associated to A . Assume for simplicity $\mathbb{Q} \subseteq R$. We consider the \mathbb{N}^I -graded R -module $\mathcal{H}_A^f(Q) := \bigoplus_{v \in \mathbb{N}^I} A_{G_v}(\text{Rep}(Q, v))$ (or simply \mathcal{H}^f when A and Q are understood from the context). For each pair of dimension vectors $v_1, v_2 \in \mathbb{N}^I$, we define the Hall multiplication

$$(2) \quad m_{v_1, v_2} : A_{G_{v_1}}^*(\text{Rep}(Q, v_1)) \otimes A_{G_{v_2}}^*(\text{Rep}(Q, v_2)) \rightarrow A_{G_v}^*(\text{Rep}(Q, v))$$

as in [KoSo11]. Start with the Künneth isomorphism

$$\otimes : A_{G_{v_1}}^*(\text{Rep}(Q, v_1)) \otimes A_{G_{v_2}}^*(\text{Rep}(Q, v_2)) \rightarrow A_{G_{v_1} \times G_{v_2}}^*(\text{Rep}(Q, v_1) \times \text{Rep}(Q, v_2)).$$

Define

$$\text{Rep}(Q)_{v_1, v_2} := \{x \in \text{Rep}(Q, v) \mid x(V_1) \subset V_1\} \subset \text{Rep}(Q, v).$$

We have the following correspondence of G_v -equivariant varieties:

$$G_v \times_P (\text{Rep}(Q, v_1) \times \text{Rep}(Q, v_2)) \xleftarrow{p} G_v \times_P (\text{Rep}(Q)_{v_1, v_2}) \xrightarrow{\eta} \text{Rep}(Q, v_1 + v_2),$$

where p is the projection, and $\eta : (g, x) \mapsto gxg^{-1}$ is the action by conjugation. Consider the following 3 morphisms:

(1) The isomorphism

$$A_{G_{v_1} \times G_{v_2}}^*(\text{Rep}(Q, v_1) \times \text{Rep}(Q, v_2)) \cong A_{G_v}^*(G_v \times_P (\text{Rep}(Q, v_1) \times \text{Rep}(Q, v_2))).$$

(2) The pullback p^* :

$$p^* : A_{G_v}^*(G_v \times_P (\text{Rep}(Q, v_1) \times \text{Rep}(Q, v_2))) \rightarrow A_{G_v}^*(G_v \times_P (\text{Rep}(Q)_{v_1, v_2})).$$

(3) The pushforward η_* :

$$\eta_* : A_{G_v}^*(G_v \times_P (\text{Rep}(Q)_{v_1, v_2})) \rightarrow A_{G_v}^*(\text{Rep}(Q, v_1 + v_2)).$$

The map m_{v_1, v_2} (2) is defined as the composition of the Künneth isomorphism with the above 3 morphisms.

Proposition 2.1. *The maps m_{v_1, v_2} for $v_1, v_2 \in \mathbb{N}^I$ fit together, defining an \mathbb{N}^I -graded associative R -algebra structure on $\mathcal{H}_A^f(Q)$.*

This is essentially Theorem 1 of [KoSo11], replacing the usual cohomology by oriented Borel-Moore homology theory A .

Definition 2.2. The \mathbb{N}^I -graded $A(\text{pt})$ -module $\mathcal{H}_A^f(Q)$ endowed with multiplication m_{v_1, v_2} is the formal cohomological Hall algebra (formal CoHA) associated to A and Q .

2.2. Formula of the formal Hall multiplication. Now let A be an oriented Borel-Moore homology theory, extended to an equivariant theory by Totaro's construction. In this subsection, we use the pushforward formula in § 1.3 to give an explicit formula of the multiplication m_{v_1, v_2} . The space $\text{Rep}(Q, v)$ is contractible. Thus, we have the isomorphism

$$A_{G_v}(\text{Rep}(Q, v)) \cong A_{G_v}(\text{pt}) \cong R[\lambda_j^i]_{i \in I, j=1, \dots, v^i}^{\mathfrak{S}_v},$$

where $R := A(\text{pt})$, and $\{\lambda_j^i\}_{j=1, \dots, v^i}$ are the Chern roots of the tautological bundle $\mathcal{R}(v^i)$. We now describe the multiplication map

$$m_{v_1, v_2} : \mathcal{H}_{v_1}^f \otimes \mathcal{H}_{v_2}^f \rightarrow \mathcal{H}_{v_1+v_2}^f.$$

It is convenient to write $\mathcal{H}_{v_1}^f$ as $R[\lambda_j^i]_{i \in I, j=1, \dots, v_1^i}^{\mathfrak{S}_{v_1}}$, and $\mathcal{H}_{v_2}^f$ as $R[\lambda_s^i]_{i \in I, s=1, \dots, v_2^i}^{\mathfrak{S}_{v_2}}$. We view $\mathcal{H}_{v_1}^f \otimes \mathcal{H}_{v_2}^f$ as a subalgebra of $R[\lambda_j^i]_{i \in I, j=1, \dots, (v_1+v_2)^i}^{\mathfrak{S}_{v_1+v_2}}$, by sending λ_s^i to λ_s^i , and λ_t^i to $\lambda_{t+v_1^i}^i$. Then, the multiplication map m_{v_1, v_2} can be considered as a map

$$m_{v_1, v_2} : R[\lambda_j^i]_{i \in I, j=1, \dots, v_1^i}^{\mathfrak{S}_{v_1}} \otimes R[\lambda_t^i]_{i \in I, t=1, \dots, v_2^i}^{\mathfrak{S}_{v_2}} \rightarrow R[\lambda_j^i]_{i \in I, j=1, \dots, v^i}^{\mathfrak{S}_v}.$$

The following formula of m_{v_1, v_2} is essentially Theorem 2 in [KoSo11].

Proposition 2.3. *For $f_i \in \mathcal{H}_{v_i}$, $i = 1, 2$, the product $m_{v_1, v_2}(f_1, f_2)$, as a symmetric function in $R[\lambda_j^i]_{i \in I, j=1, \dots, (v_1+v_2)^i}^{\mathfrak{S}_{v_1+v_2}}$, is given by the following formula:*

$$\sum_{\sigma \in \text{Sh}(v_1, v_2)} \sigma \left(f_1(\lambda_s^i) f_2(\lambda_t^i) \frac{\prod_{i, j \in I} \prod_{s=1}^{v_1^i} \prod_{t=1}^{v_2^i} (\lambda_t^j -_F \lambda_s^i)^{a_{ij}}}{\prod_{i \in I} \prod_{s=1}^{v_1^i} \prod_{t=1}^{v_2^i} (\lambda_t^i -_F \lambda_s^i)} \right),$$

where a_{ij} is the number of arrows from vertex i to vertex j .

Proof. Let $i : \text{Rep}(Q)_{v_1, v_2} \hookrightarrow \text{Rep}(Q, v)$ be the embedding. The pushforward η_* is the composition of the following two morphisms:

$$\begin{aligned} i_* &: A_P(\text{Rep}(Q, v)_{v_1, v_2}) \rightarrow A_P(\text{Rep}(Q, v)) \\ \pi_* &: A_P(\text{Rep}(Q, v)) \cong A_{G_v}(G_v \times_P \text{Rep}(Q, v)) \rightarrow A_{G_v}(\text{Rep}(Q, v)). \end{aligned}$$

The pushforward i_* in the equivariant oriented Borel-Moore theory is given by $i_*(f) = f \cdot e_{v_1, v_2}$, where e_{v_1, v_2} is the equivariant Euler class of the normal bundle of i . The embedding i induces the following embedding of vector bundles on $BL := \text{Grass}(v_1, \infty) \times \text{Grass}(v_2, \infty)$:

$$\text{Rep}(Q)_{v_1, v_2} \times_{(G_{v_1} \times G_{v_2})} (EG_{v_1} \times EG_{v_2}) \hookrightarrow \text{Rep}(Q, v) \times_{(G_v \times G_v)} (EG_v \times EG_v).$$

By definition, e_{v_1, v_2} is the Euler class of the quotient bundle. We identify the quotient bundle with

$$\bigoplus_{h \in H} \mathcal{H}om_{\mathcal{O}}(\mathcal{R}(v_1^{\text{out}(h)}), \mathcal{R}(v_2^{\text{in}(h)})) \cong \bigoplus_{i, j \in I} (\mathcal{R}(v_1^i)^* \otimes \mathcal{R}(v_2^j))^{a_{ij}},$$

where $\mathcal{R}(r)$ is the tautological bundle of $\text{Grass}(r, \infty)$. Thus, the equivariant Euler class is

$$e_{v_1, v_2} = \prod_{i, j \in I} \prod_{s=1}^{v_1^i} \prod_{t=1}^{v_2^j} (\lambda_t^j -_F \lambda_s^i)^{a_{ij}}.$$

As a consequence, the multiplication map m_{v_1, v_2} sends $f_1 \in \mathcal{H}_{v_1}^f$ and $f_2 \in \mathcal{H}_{v_2}^f$ to $\text{pr}_*(f_1 \cdot f_2 \cdot e_{v_1, v_2})$, where pr is the projection $\text{Grass}(v_1, \mathcal{R}(v)) \rightarrow \text{Gr}(v, \infty)$. Applying Proposition 1.6 to pr , we get the multiplication formula. \square

Example 2.4. Let Q be a quiver with one single vertex and a loops. The formal cohomological Hall algebra is $\mathcal{H}^f = \bigoplus_n \mathcal{H}_n^f$ with $\mathcal{H}_n^f = \mathbb{Q}[\lambda_1, \dots, \lambda_n]^{\mathfrak{S}_n}$. The Hall multiplication of $f_1 \in \mathcal{H}_r^f$ and $f_2 \in \mathcal{H}_{n-r}^f$ becomes

$$\begin{aligned} m(f_1, f_2) &= \sum_{\{J \subset [1, \dots, n], |J|=r\}} \frac{f_1(\lambda_j) f_2(\lambda_i)_{j \in J, i \notin J}}{\prod_{j \in J, i \notin J} (\lambda_i -_F \lambda_j)} \cdot \prod_{j \in J, i \notin J} (\lambda_i -_F \lambda_j)^a \\ &= \sum_{\{\sigma \in \text{Sh}(r, n-r)\}} \sigma \cdot \left(\frac{f_1(\lambda_1, \dots, \lambda_r) f_2(\lambda_{r+1}, \dots, \lambda_n)}{\prod_{1 \leq j \leq r, (r+1) \leq i \leq n} (\lambda_i -_F \lambda_j)} \prod_{1 \leq j \leq r, (r+1) \leq i \leq n} (\lambda_i -_F \lambda_j)^a \right). \end{aligned}$$

3. THE GENERALIZED SHUFFLE ALGEBRAS

Let (R, F) be any formal group law. Again, assume $\mathbb{Q} \subseteq R$. In this section, we define the generalized formal shuffle algebra \mathcal{SH} associated to the formal group law (R, F) and the quiver Q . In the shuffle algebra considered in this section, there are two quantization parameters t_1, t_2 . Geometrically these two quantization parameters come from the two dimensional torus $T = \mathbb{G}_m^2$ action on the cotangent bundle of representation space of the quiver Q .

3.1. The formal shuffle algebra. The formal shuffle algebra \mathcal{SH} is a \mathbb{N}^I -graded $R[[t_1, t_2]]$ -algebra. As a $R[[t_1, t_2]]$ -module, we have $\mathcal{SH} = \bigoplus_{v \in \mathbb{N}^I} \mathcal{SH}_v$. The degree v piece is

$$\mathcal{SH}_v := R[[t_1, t_2]] [\lambda_s^i]_{i \in I, s=1, \dots, v^i}^{\mathfrak{S}_v}.$$

For any v_1 and $v_2 \in \mathbb{N}^I$, we consider $\mathcal{SH}_{v_1} \otimes \mathcal{SH}_{v_2}$ as a subalgebra of

$$R[[t_1, t_2]] [\lambda_j^i]_{i \in I, j=1, \dots, (v_1+v_2)^i}$$

by sending λ_s^i to λ_s^i , and λ_t^i to $\lambda_{t+v_1^i}^i$. Set:

$$(3) \quad \text{fac}_1 := \prod_{i \in I} \prod_{s=1}^{v_1^i} \prod_{t=1}^{v_2^i} \frac{\lambda_s^i -_F \lambda_t^i +_F t_1 +_F t_2}{\lambda_t^i -_F \lambda_s^i}.$$

For each arrow $h \in H$, associate two integers, m_h and m_{h^*} . Define

$$(4) \quad \text{fac}_2 := \prod_{h \in H} \left(\prod_{s=1}^{v_1^{\text{out}(h)}} \prod_{t=1}^{v_2^{\text{in}(h)}} (\lambda_t^{\text{in}(h)} -_F \lambda_s^{\text{out}(h)} +_F m_h \cdot t_1) \prod_{s=1}^{v_1^{\text{in}(h)}} \prod_{t=1}^{v_2^{\text{out}(h)}} (\lambda_t^{\text{out}(h)} -_F \lambda_s^{\text{in}(h)} +_F m_{h^*} \cdot t_2) \right),$$

where $m \cdot t = t +_F t +_F \dots +_F t$ is the summation of m terms, for $m \in \mathbb{N}$. When $m_h = m_{h^*} = 1$, the fac_2 in (4) can be simplified as

$$\text{fac}_2 = \prod_{i, j \in I} \prod_{s=1}^{v_1^i} \prod_{t=1}^{v_2^j} (\lambda_t^j -_F \lambda_s^i +_F t_1)^{a_{ij}} (\lambda_t^j -_F \lambda_s^i +_F t_2)^{a_{ji}},$$

where a_{ij} is the number of arrows from i to j of quiver Q .

The multiplication of $f_1(\lambda') \in \mathcal{SH}_{v_1}$ and $f_2(\lambda'') \in \mathcal{SH}_{v_2}$ is defined to be

$$(5) \quad \sum_{\sigma \in \text{Sh}(v_1, v_2)} \sigma(f_1 \cdot f_2 \cdot \text{fac}_1 \cdot \text{fac}_2) \in R[[t_1, t_2]] \llbracket \lambda_j^i \rrbracket_{i \in I, j=1, \dots, (v_1+v_2)}^{\mathfrak{S}_{v_1+v_2}}.$$

3.2. The geometric construction of the generalized shuffle algebra. With notations as before, let $Q = (I, H)$ be a quiver. Let $\overline{Q} = Q \sqcup Q^{\text{op}}$ be the double of Q . That is, \overline{Q} has the same vertex set as Q and whose set of arrows is a disjoint union of the sets of arrows of Q and Q^{op} , an opposite quiver. To be more precise, the set of arrows of \overline{Q} is $H \sqcup H^{\text{op}}$. There is a bijection $H \rightarrow H^{\text{op}}$, such that, for each $h \in H$, there is a reverse arrow $h^* \in H^{\text{op}}$, with $\text{out}(h^*) = \text{in}(h)$ and $\text{in}(h^*) = \text{out}(h)$. We have the following isomorphisms

$$\text{Rep}(\overline{Q}, v) \cong \text{Rep}(Q, v) \times \text{Rep}(Q^{\text{op}}, v) \cong T^* \text{Rep}(Q, v).$$

The algebraic group G_v acts on $T^* \text{Rep}(Q, v)$ by conjugation. The torus $T = \mathbb{G}_m^2$ acts on $T^* \text{Rep}(Q, v)$ the same way as in [Nak01, (2.7.1) and (2.7.2)]. Let a be the number of arrows in Q from vertex i to j . We fix a numbering h_1, \dots, h_a of the arrows in Q . The corresponding reversed arrows in Q^{op} are labelled by h_1^*, \dots, h_a^* . For $h_p \in H$, and $B = (B_p) \in \text{Hom}(V^i, V^j)$, $B^* = (B_p^*) \in \text{Hom}(V^j, V^i)$, Define the $T = \mathbb{G}_m^2$ -action by:

$$t_1 \cdot B_p := t_1^{m_{h_p}} B_p, \quad t_2 \cdot B_p^* := t_2^{m_{h_p^*}} B_p^*.$$

We assume the T -action on $T^* \text{Rep}(Q, v)$ satisfies the following.

Assumption 3.1. For any $h \in H$, $t_1^{m(h)} t_2^{m(h^*)}$ is a constant, i.e., does not depend on $h \in H$.

One example of T action satisfying Assumption 3.1 is the following.

Remark 3.2. The T factors through \mathbb{G}_m , i.e., $t_1 = t_2 = \hbar/2$, and for any pair of vertices i and j with arrows h_1, \dots, h_a from i to j , the pairs of integers are $m_{h_p} = a + 2 - 2p$ and $m_{h_p^*} = -a + 2p$. This choice is essential in § 6.

For any pair of dimension vectors $v_1, v_2 \in \mathbb{N}^I$, we consider a map

$$m_{v_1, v_2}^S : A_{G_{v_1} \times T}(\text{Rep}(\overline{Q}, v_1)) \otimes_{R[[t_1, t_2]]} A_{G_{v_2} \times T}(\text{Rep}(\overline{Q}, v_2)) \rightarrow A_{G_{v_1+v_2} \times T}(\text{Rep}(\overline{Q}, v_1 + v_2)),$$

defined as follows. Let $v = v_1 + v_2$. In the Lagrangian correspondence formalism in § 1.4, we take Y to be $\text{Rep}(Q, v_1) \times \text{Rep}(Q, v_2)$, X' to be $\text{Rep}(Q, v_1 + v_2)$, and \mathcal{V} to be $\text{Rep}(Q)_{v_1, v_2}$. Recall that

$$\text{Rep}(Q)_{v_1, v_2} := \{x \in \text{Rep}(Q, v) \mid x(V_1) \subset V_1\} \subset \text{Rep}(Q, v).$$

We write $G := G_v$, and $P \subset G_v$, the parabolic subgroup preserving the subspace V_1 . Let $L := G_{v_1} \times G_{v_2}$ be the Levi subgroup of P .

As in § 1.4, we have the following Lagrangian correspondence of $G \times T$ -varieties:

$$G \times_P T^* Y \xrightarrow{\iota} T^*(G \times_P Y) \xleftarrow{\phi} Z \xrightarrow{\psi} \text{Rep}(\overline{Q}, v_1 + v_2).$$

We now define the multiplication map m_{v_1, v_2}^S . We first have the Künneth isomorphism.

$$\otimes : A_{G_{v_1} \times T}(\text{Rep}(\overline{Q}, v_1)) \otimes_{R[[t_1, t_2]]} A_{G_{v_2} \times T}(\text{Rep}(\overline{Q}, v_2)) \cong A_{G_{v_1} \times G_{v_2} \times T}(\text{Rep}(\overline{Q}, v_1) \times \text{Rep}(\overline{Q}, v_2)).$$

Consider the following sequence of morphisms:

- (1) The isomorphism: $A_{G_{v_1} \times G_{v_2} \times T}(\text{Rep}(\overline{Q}, v_1) \times \text{Rep}(\overline{Q}, v_2)) \cong A_{G \times T}(G \times_P T^* Y)$.
- (2) The pushforward map: $\iota_* : A_{G \times T}(G \times_P T^* Y) \rightarrow A_{G \times T}(T^*(G \times_P Y))$.

(3) Following the notations in the Lagrangian correspondence diagram, we have

$$A_{G \times T}(T^*(G \times_P Y)) \xrightarrow{\phi^*} A_{G \times T}(Z) \xrightarrow{\psi_*} A_{G \times T}(\text{Rep}(\overline{Q}, v)).$$

Note that ψ is a proper morphism, and hence the push-forward is well-defined.

We define map m_{v_1, v_2}^S to be the composition of the Künneth isomorphism with the above sequence of 3 morphisms.

Proposition 3.3. *The multiplication maps m_{v_1, v_2}^S are associative.*

Proof. The proof follows the same idea as in [SV12, Proposition 7.5]. For the convenience of the readers, we include a proof here. Fix a flag $V_1 \subset V_2 \subset V$, where V_i is an I -tuple subspaces of V of dimension vector $v_1 + \dots + v_i$. Let P_1, P_{12} be the parabolic subgroups $P_1 := \{g \in G_v | g(V_1) \subset V_1\}$, and $P_{12} := \{g \in G_v | g(V_1) \subset V_1, g(V_2) \subset V_2\}$.

We first define the following varieties.

- Let X_1 be the set of quadruples (F_1, F_2, a) , where $F_1 \subset F_2 \subset V$ is a flag such that $F_1 \cong V_1, F_2 \cong V_2$, and $a \in \text{Rep}(Q, v)$ is an endomorphism of the vector space $F_1 \oplus (F_2/F_1) \oplus (V/F_2)$.
- Let X_2 be the set of pairs (F_1, a) , where $F_1 \subset V$, such that $F_1 \cong V_1$ and $a \in \text{Rep}(Q, v)$ is an endomorphism of the vector space $F_1 \oplus (V/F_1)$.
- $X_3 = \text{Rep}(Q, v)$.

We then define the following correspondences. For $i = 1, 2, 3$, let W_i be the following

$$W_1 = \{(F_1, a) \mid F_1 \subset V, \text{ such that } F_1 \cong V_1, \text{ and } a(F_1) \subset F_1, \text{ for } a \in \text{Rep}(Q, v)\}.$$

$$W_2 = \{(F_1, F_2, a) \mid F_1 \subset F_2 \subset V, a \in \text{Rep}(Q, v), \text{ such that } F_j \cong V_j, \text{ and } a(F_j) \subset F_j, \text{ for } j = 1, 2\}.$$

$$W_3 = \{(F_1, F_2, a) \mid F_1 \subset F_2 \subset V, a \in \text{Rep}(Q, v), \text{ such that } F_j \cong V_j, \text{ for } j = 1, 2, \text{ and } a \in \text{End}(F_1 \oplus V/F_1), a \text{ preserves the subspace } \{0\} \oplus F_2/F_1\}.$$

Let Z_i be the conormal bundle of W_i . Consider the following commutative diagram with the square being Cartesian.

$$\begin{array}{ccccc} T^*X_3 & \xleftarrow{\psi_1} & Z_1 & \xrightarrow{\phi_1} & T^*X_2 \\ & \searrow \psi_2 & \uparrow & & \uparrow \psi_3 \\ & & Z_2 & \xrightarrow{\quad} & Z_3 \\ & & & \searrow \phi_2 & \downarrow \phi_3 \\ & & & & T^*X_1. \end{array}$$

By Lemma 1.14 and Lemma 1.16, we have $I_2 = I_1 \circ I_3$, where

$$I_1 = \psi_{1*} \circ \phi_1^* : A_{G \times T}(T^*X_2) \rightarrow A_{G \times T}(T^*X_3).$$

$$I_2 = \psi_{2*} \circ \phi_2^* : A_{G \times T}(T^*X_1) \rightarrow A_{G \times T}(T^*X_3).$$

$$I_3 = \psi_{3*} \circ \phi_3^* : A_{G \times T}(T^*X_1) \rightarrow A_{G \times T}(T^*X_2).$$

An argument similar to [L91, Lemma 3.4] implies the associativity. \square

Now assume A is an arbitrary oriented Borel-Moore homology theory, extended to an equivariant Borel-Moore homology by Totaro's construction. For any $v \in \mathbb{N}^I$, we identify

$$\mathcal{SH}_v := R[[t_1, t_2]] \llbracket \lambda_s^i \rrbracket_{i \in I, s=1, \dots, v^i}^{\mathfrak{S}_v}$$

with the $A_T(\text{pt})$ -module $A_{G_v \times T}(\text{Rep}(\overline{Q}, v))$. Such identification comes from the extended homotopy equivalence property of A , i.e.,

$$A_{G_v \times T}(\text{Rep}(\overline{Q}, v)) \cong A_T(\text{pt}) \otimes A(BG_v) \cong R[[t_1, t_2]] [\lambda_s^i]_{i \in I, s=1, \dots, v^i}^{\mathfrak{S}_v}.$$

We identify the algebraically defined formal shuffle algebra multiplication in § 3.1 with the geometrically defined m_{v_1, v_2}^S .

Proposition 3.4. *Under the identification*

$$\mathcal{SH}_v := R[[t_1, t_2]] [\lambda_s^i]_{i \in I, s=1, \dots, v^i}^{\mathfrak{S}_v} \cong A_{G_v \times T}(\text{Rep}(\overline{Q}, v)),$$

the map m_{v_1, v_2}^S is equal to the multiplication map (5) of the shuffle algebra.

Proof. Let e_{v_1, v_2}^ι be the equivariant Euler class of the normal bundle of the embedding

$$\iota : G_v \times_{G_{v_1, v_2}} T^*Y \hookrightarrow T^*(G_v \times_{G_{v_1, v_2}} Y).$$

The normal bundle of ι is isomorphic to T^*G/P as a bundle over the Grassmannian G/P . Also T^*G/P is in turn isomorphic to $\bigoplus_{i \in I} (\mathcal{R}(v_1^i) \otimes \mathcal{R}(v_2^i)^*)$. Thus we have

$$e_{v_1, v_2}^\iota = \prod_{i \in I} \prod_{s=1}^{v_1^i} \prod_{t=1}^{v_2^i} (\lambda_s^i -_F \lambda_t^{i'} +_F t_1 +_F t_2).$$

Therefore, ι_* is multiplication by e_{v_1, v_2}^ι .

The composition $Z := T_W^*(X \times X') \rightarrow W \rightarrow G/P$ is a vector bundle, where the second map is the natural projection. It induces a morphism $EG \times_G Z \rightarrow EG \times_G (G/P) \cong BL$. Recall that

$$BL \cong \text{Gr}(v_1, \infty) \times \text{Gr}(v_2, \infty) \cong EG/P \xrightarrow{p} BG.$$

Note that $EG \times_G T^* \text{Rep}(Q, v)$ is a vector bundle over BG . Let $p^* T^* \text{Rep}(Q, v)$ be the pull-back vector bundle on BL via the projection $p : BL \rightarrow BG$. The natural map $\psi : EG \times_G Z \rightarrow EG \times_G T^* \text{Rep}(Q, v)$ factors through $\psi_1 : EG \times_G Z \rightarrow p^* T^* \text{Rep}(Q, v)$ by the universality of the pullback. We summarize these notations in the following diagram

$$\begin{array}{ccccc} EG \times_G Z & \xrightarrow{\psi} & EG \times_G T^* \text{Rep}(Q, v) & & \\ \downarrow \psi_1 & \searrow p' & \downarrow \pi' & & \\ p^* T^* \text{Rep}(Q, v) & \xrightarrow{p'} & EG \times_G T^* \text{Rep}(Q, v) & & \\ \downarrow \pi & & \downarrow \pi' & & \\ BL & \xrightarrow{p} & BG = \text{Grass}(v, \infty). & & \end{array}$$

The pushforward map ψ_* is the composition $p'_* \circ \psi_{1*}$.

The map $\psi_1 : EG \times_G Z \hookrightarrow p^* T^* \text{Rep}(Q, v)$ is an embedding of vector bundles on BG . The pushforward ψ_{1*} is the multiplication by the equivariant Euler class $e_{v_1, v_2}^{\psi_1}$ of the normal bundle of the embedding ψ_1 . The normal bundle to ψ_1 can be identified with

$$\begin{aligned} & \mathcal{H}om_{\overline{Q}}(\mathcal{R}(v_1), \mathcal{R}(v_2)) \\ &= \bigoplus_{h \in H} \mathcal{H}om_{\mathcal{O}}(\mathcal{R}(v_1^{\text{out}(h)}), \mathcal{R}(v_2^{\text{in}(h)})) \bigoplus \bigoplus_{h \in H^{\text{op}}} \mathcal{H}om_{\mathcal{O}}(\mathcal{R}(v_1^{\text{out}(h)}), \mathcal{R}(v_2^{\text{in}(h)})) \end{aligned}$$

over $BL = \text{Grass}(v_1, \infty) \times \text{Grass}(v_2, \infty)$. Here, the first copy $\bigoplus_{h \in H} \mathcal{H}\text{om}_{\mathcal{O}}(\mathcal{R}(v_1^{\text{out}(h)}), \mathcal{R}(v_2^{\text{in}(h)}))$ is considered as a subspace of $\text{Rep}(Q, v)$, and the second copy $\bigoplus_{h \in H^{\text{op}}} \mathcal{H}\text{om}_{\mathcal{O}}(\mathcal{R}(v_1^{\text{out}(h)}), \mathcal{R}(v_2^{\text{in}(h)}))$ as a subspace of $\text{Rep}(Q^{\text{op}}, v)$. Thus, the equivariant Euler class of the normal bundle to ψ_1 is

$$e_{\psi_1, v_2}^{\psi_1} = \prod_{h \in H} \left(\prod_{s=1}^{v_1^{\text{out}(h)}} \prod_{t=1}^{v_2^{\text{in}(h)}} (\lambda_t^{\text{in}(h)} -_F \lambda_s^{\text{out}(h)} +_F m_h \cdot t_1) \prod_{s=1}^{v_1^{\text{in}(h)}} \prod_{t=1}^{v_2^{\text{out}(h)}} (\lambda_t^{\text{out}(h)} -_F \lambda_s^{\text{in}(h)} +_F m_h \cdot t_2) \right).$$

So far, we obtained that ι_* is multiplication by $e_{v_1, v_2}^{\psi_1}$, and ψ_{1*} is multiplication by $e_{v_1, v_2}^{\psi_1}$. The map p' is the pull-back of p via the projection π' . Therefore, p' is also a Grassmannian bundle, and consequently p'_* is given by Proposition 1.6. Putting all the above together, the map m_{v_1, v_2}^S is given by exactly the same formula as (5). \square

Remark 3.5. When A is the equivariant Chow group, under the identification,

$$\mathcal{SH}_v := R[t_1, t_2][\lambda_s^i]_{i \in I, s=1, \dots, v_i}^{\mathfrak{S}_v} \cong A_{G_v \times T}(\text{Rep}(\overline{Q}, v)),$$

Theorem 3.4 still holds. Similar for the equivariant K -theory, which will be spelled out in details below.

3.3. An example: K -theory shuffle algebra. As an example, we take A to be the K -theory with rational coefficients. We relate the shuffle algebra \mathcal{SH} with the Feigin-Odesskii shuffle algebra (see [FO97]) in the Jordan quiver case.

For any line bundle $p : L \rightarrow X$ on a smooth quasi-projective variety X . Let $s : X \rightarrow L$ be the zero-section. The resolution of $s_* \mathcal{O}_X$:

$$1 \rightarrow p^* L^\vee \rightarrow \mathcal{O}_L \rightarrow s_* \mathcal{O}_X \rightarrow 0$$

implies that the first Chern class of L in the K -theory is $c_1(L) := s^* s_* (\mathcal{O}_X) = 1 - L^\vee$. As a consequence, the formal group law $(K(\text{pt}), F_m)$ is $F_m(u, v) = u +_{F_m} v = u + v - uv$. In particular, $u -_{F_m} v = \frac{u-v}{1-v}$.

For $r \in \mathbb{N}$, let $\mathcal{R}(r)$ be the tautological vector bundle of $\text{Grass}(r, \infty)$ and let $\text{Fl}(\mathcal{R}(r))$ be the associate full flag bundle. We identify $K(\text{Grass}(r, \infty))$ with $\mathbb{Q}[z_1^\pm, \dots, z_r^\pm]^{\mathfrak{S}_r}$ with z_i being the i -th tautological line bundle (rather than its 1st Chern class) of $\text{Fl}(\mathcal{R}(r))$. Let s_i be the one-dimensional natural representation of the i -th copy of \mathbb{G}_m in T . Thus, we have $t_i = 1 - \frac{1}{s_i}$, for $i = 1, 2$. By Theorem 3.4, we get the following.

Corollary 3.6. *Let Q be any quiver, and A be the K -theory with rational coefficients. Assume $m_h = m_{h^*} = 1$ for any $h \in H$. For any $v \in \mathbb{N}^I$, identify \mathcal{SH}_v with $\mathbb{Q}[(z_1^i)^\pm, \dots, (z_{v_i}^i)^\pm]_{i \in I}^{\mathfrak{S}_v}$ as above. For any pair of dimension vectors $v_1, v_2 \in \mathbb{N}^I$, and $f_i \in \mathcal{SH}_{v_i}$ for $i = 1, 2$, $m_{v_1, v_2}(f_1 \otimes f_2)$ is equal to*

$$\sum_{\sigma \in \text{Sh}(v_1, v_2)} \sigma \left(f_1(z_s^i) f_2(z_t^i) \prod_{i \in I} \prod_{s=1}^{v_1^i} \prod_{t=1}^{v_2^i} \frac{1 - \frac{z_t^i}{z_s^i s_1 s_2}}{1 - \frac{z_s^i}{z_t^i s_1}} \cdot \prod_{i, j \in I} \prod_{s=1}^{v_1^i} \prod_{t=1}^{v_2^j} (1 - \frac{z_s^i}{z_t^j s_1})^{a_{ij}} (1 - \frac{z_s^i}{z_t^j s_2})^{a_{ji}} \right).$$

Example 3.7. Let Q be the Jordan quiver and A be the K -theory with rational coefficients. Then as a vector space, $\mathcal{SH} \cong \bigoplus_n \mathbb{Q}[s_1^\pm, s_2^\pm][z_1^\pm, \dots, z_n^\pm]^{\mathfrak{S}_n}$. The multiplication $\mathcal{SH}_r \otimes \mathcal{SH}_{n-r} \rightarrow \mathcal{SH}_n$ sends $f_1(z_1, \dots, z_r) \otimes f_2(z_{r+1}, \dots, z_n)$ to

$$\sum_{\sigma \in \text{Sh}(r, n-r)} \sigma \cdot \left(f_1 \cdot f_2 \cdot \prod_{1 \leq j \leq r, (r+1) \leq i \leq n} \frac{(1 - \frac{z_j}{z_i s_1 s_2})(1 - \frac{z_i}{z_j s_1})(1 - \frac{z_i}{z_j s_2})}{1 - \frac{z_i}{z_j}} \right).$$

Let \mathcal{SH}' be the shuffle algebra defined by Feigin-Odesskii in [FO97]. (See also [FT09].) By definition, $\mathcal{SH}' \cong \bigoplus \mathbb{Q}[q_1^\pm, q_2^\pm][z_1^\pm, \dots, z_n^\pm]^{\mathfrak{S}_n}$, with multiplication $f_1(z_1, \dots, z_r) \otimes f_2(z_{r+1}, \dots, z_n)$

$$\mapsto \sum_{\{\sigma \in \text{Sh}(r, n-r)\}} \sigma \cdot \left(f_1 \cdot f_2 \cdot \prod_{1 \leq j \leq r, (r+1) \leq i \leq n} \frac{(1 - q_1 z_i / z_j)(1 - q_2 z_i / z_j)}{(1 - z_i / z_j)(1 - q_1 q_2 z_i / z_j)} \right).$$

Define an algebra homomorphism $\mathcal{SH}' \rightarrow \mathcal{SH}$

$$\begin{aligned} q_i &\mapsto s_i^{-1}, \text{ for } i = 1, 2; \\ f(z_1, \dots, z_n) &\mapsto f(z_1, \dots, z_n) Y_n, \text{ for } n \in \mathbb{N}, \end{aligned}$$

where

$$Y_n = \prod_{1 \leq j < i \leq n} \left((1 - q_1 q_2 \frac{z_j}{z_i})(1 - q_1 q_2 \frac{z_i}{z_j}) \right).$$

The map is well-defined, since the factor Y_n is invariant under the action of $\text{Sh}(r, n-r)$. A straightforward calculation shows that this is an algebra homomorphism.

When Q is the cyclic quiver, and A is the K -theory, it is shown in [Ne15] that the shuffle algebra is isomorphic to the positive half of the quantum toroidal algebra of type A .

4. THE PREPROJECTIVE COHOMOLOGICAL HALL ALGEBRAS

In this section, we introduce the main object to be studied in this paper, the preprojective CoHA. The representations of this algebra are realized as A -homology of Nakajima quiver varieties. Now we describe the multiplication of this algebra.

4.1. Hall multiplication. Notations are as before. Let $Q = (I, H)$ be a quiver, and let $v \in \mathbb{N}^I$ be a dimension vector. The group $G_v := \prod_{i \in I} \text{GL}_{v_i}$ acts on the cotangent space $T^* \text{Rep}(Q, v)$ via conjugation. Let \mathfrak{g}_v be the Lie algebra of G_v . Let

$$\mu_v : T^* \text{Rep}(Q, v) \rightarrow \mathfrak{g}_v^*, \quad (x, x^*) \mapsto [x, x^*]$$

be the moment map. Note that the closed subvariety $\mu_v^{-1}(0) \subset T^* \text{Rep}(Q, v)$ could be singular in general.

We consider the \mathbb{N}^I -graded $R[[t_1, t_2]]$ -module ${}^A \mathcal{P}(Q) := \bigoplus_v {}^A \mathcal{P}_v(Q)$ with ${}^A \mathcal{P}_v(Q) = A_{G_v \times T}(\mu_v^{-1}(0))$. When both A and Q are understood from the context, we will use the abbreviations \mathcal{P} and \mathcal{P}_v . For each pair $v_1, v_2 \in \mathbb{N}^I$, we define the multiplication map $m_{v_1, v_2}^P : \mathcal{P}_{v_1} \otimes_{R[[t_1, t_2]]} \mathcal{P}_{v_2} \rightarrow \mathcal{P}_{v_1 + v_2}$.

We write $v = v_1 + v_2$. We consider the Lagrangian correspondence formalism in § 1.4, with the following specializations: Take Y to be $\text{Rep}(Q, v_1) \times \text{Rep}(Q, v_2)$, X' to be $\text{Rep}(Q, v)$ and \mathcal{V} to be $\text{Rep}(Q)_{v_1, v_2}$. As in § 3.2, we write $G := G_v$ for short. Let $P \subset G_v$ be the parabolic subgroup and $L := G_{v_1} \times G_{v_2}$ be the Levi subgroup of P . Recall in § 1.4, we have the following correspondence of $G \times T$ -varieties:

$$\begin{array}{ccccccc} G \times_P (\mu_{v_1}^{-1}(0) \times \mu_{v_2}^{-1}(0)) & \xlongequal{\quad} & T_G^* X & \xleftarrow{\bar{\phi}} & Z_G & \xrightarrow{\bar{\psi}} & \mu_v^{-1}(0) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G \times_P T^* Y & \xrightarrow{\iota} & T^*(G \times_P Y) & \xleftarrow{\phi} & Z & \xrightarrow{\psi} & \text{Rep}(\bar{Q}, v). \end{array}$$

We have the Künneth morphism (which may or may not be an isomorphism).

$$\otimes : \mathcal{P}_{v_1} \otimes_{R[[t_1, t_2]]} \mathcal{P}_{v_2} \rightarrow A_{G_{v_1} \times G_{v_2} \times T}(\mu_{v_1}^{-1}(0) \times \mu_{v_2}^{-1}(0)).$$

Consider the following sequence of morphisms:

- (1) The natural projection $G_{v_1} \times G_{v_2} \leftarrow P$ is homotopy equivalence. It induces the following isomorphism $A_{G_{v_1} \times G_{v_2} \times T}(\mu_{v_1}^{-1}(0) \times \mu_{v_2}^{-1}(0)) \cong A_{P \times T}(\mu_{v_1}^{-1}(0) \times \mu_{v_2}^{-1}(0))$. We have the following isomorphism

$$A_{P \times T}(\mu_{v_1}^{-1}(0) \times \mu_{v_2}^{-1}(0)) \cong A_{G \times T}(G \times_P (\mu_{v_1}^{-1}(0) \times \mu_{v_2}^{-1}(0))).$$

- (2) Following the Lagrangian correspondence diagram, we have

$$A_{G \times T}(T_G^* X) \xrightarrow{\phi^\sharp} A_{G \times T}(Z_G) \xrightarrow{\bar{\psi}_*} A_{G \times T}(\mu_v^{-1}(0)) \cong \mathcal{P}_v,$$

where ϕ^\sharp is the Gysin pullback of ϕ .

The map m_{v_1, v_2}^P is defined to be the composition of the above morphisms.

Theorem 4.1. *The maps m_{v_1, v_2}^P fit together to define an associative algebra structure on \mathcal{P} .*

Proof. We keep the same notations as in the proof of Proposition 3.3. By definition, $T_G^* X_3 = \mu_v^{-1}(0)$. By Lemma 1.10, $T_G^* X_2 = G_v \times_{P_1} (\mu_{v_1}^{-1}(0) \times \mu_{v_2+v_3}^{-1}(0))$ and

$$T_G^* X_1 = G_v \times_{P_{12}} (\mu_{v_1}^{-1}(0) \times \mu_{v_2}^{-1}(0) \times \mu_{v_3}^{-1}(0)).$$

By Lemma 1.14 and Lemma 1.16, we have $\bar{I}_2 = \bar{I}_1 \circ \bar{I}_3$, where

$$\begin{aligned} \bar{I}_1 &= \bar{\psi}_{1*} \circ \phi_1^\sharp : A_{G \times T}(T_G^* X_2) \rightarrow A_{G \times T}(T_G^* X_3). \\ \bar{I}_2 &= \bar{\psi}_{2*} \circ \phi_2^\sharp : A_{G \times T}(T_G^* X_1) \rightarrow A_{G \times T}(T_G^* X_3). \\ \bar{I}_3 &= \bar{\psi}_{3*} \circ \phi_3^\sharp : A_{G \times T}(T_G^* X_1) \rightarrow A_{G \times T}(T_G^* X_2). \end{aligned}$$

An argument similar to [L91, Lemma 3.4] implies the associativity of the multiplication m^P . \square

Definition 4.2. For any $v \in \mathbb{N}^I$, consider the \mathbb{N}^I -graded $R[[t_1, t_2]]$ -module $\mathcal{P} := \bigoplus_v \mathcal{P}_v$, where $\mathcal{P}_v := A_{G_v \times T}(\mu_v^{-1}(0))$. The preprojective cohomological Hall algebra (CoHA) of the quiver Q is the associative algebra \mathcal{P} endowed with the Hall multiplication m_{v_1, v_2}^P .

The name preprojective CoHA is motivated by the fact that the subvariety $\mu_v^{-1}(0) \subset \text{Rep}(\bar{Q}, v)$ parametrizes representations of the preprojective algebra.

Theorem 4.3. *There is a well-defined morphism of $R[[t_1, t_2]]$ -algebras*

$${}^A\mathcal{P} \rightarrow \mathcal{SH}$$

induced from the embedding $i_v : \mu_v^{-1}(0) \hookrightarrow \text{Rep}(\bar{Q}, v)$.

Proof. The pushforward i_* induces a well-defined morphism

$$i_{v*} : A_{G \times T}(\mu_v^{-1}(0)) \rightarrow A_{G \times T}(\text{Rep}(\bar{Q}, v)) \cong \mathcal{SH}_v.$$

According to Lemma 1.12, it is an algebra homomorphism. \square

Remark 4.4. The algebra homomorphism in Theorem 4.3 becomes an isomorphism after localization. This follows from the Thomason localization theorem and the fact that $\mu_v^{-1}(0)$ has only one T -fixed point.

4.2. Spherical subalgebras. In general, the algebra \mathcal{P} defined above and the shuffle algebra \mathcal{SH} in § 3 are different, but closely related. Each one has a spherical subalgebra. Conjecturally, their spherical subalgebras are isomorphic, but at present, this is not known.

For any vertex $k \in I$ of the quiver Q , let e_k be the dimension vector such that $e_k^i = \delta_{ki}$. In other words, e_k has value 1 on vertex k , and 0 otherwise.

Definition 4.5. The spherical subalgebra ${}^A\mathcal{P}^s(Q) \subset {}^A\mathcal{P}(Q)$ is the subalgebra generated by \mathcal{P}_{e_k} for $k \in I$. We will abbreviate as ${}^A\mathcal{P}^s$ or \mathcal{P}^s if understood. Similarly, define $\mathcal{SH}^s \subset \mathcal{SH}$ to be the subalgebra of \mathcal{SH} generated by \mathcal{SH}_{e_k} for $k \in I$.

Proposition 4.6. *The morphism in Theorem 4.3 restricts to a surjective morphism on the spherical subalgebras*

$$\mathcal{P}^s \rightarrow \mathcal{SH}^s.$$

Proof. The surjectivity of the restriction $\mathcal{P}^s \rightarrow \mathcal{SH}^s$ follows from the isomorphism $\mathcal{P}_{e_k} \cong \mathcal{SH}_{e_k}$, for $k \in I$. Here $\mathcal{P}_{e_k} \cong A_{G_{e_k} \times T}(\text{pt}) = A_T(\text{pt})[[z_k]]$ and $\mathcal{SH}_{e_k} \cong A_T(\text{pt})[[\lambda^k]]$. \square

5. REPRESENTATIONS OF THE PREPROJECTIVE COHA

We construct representations of ${}^A\mathcal{P}(Q)$ in this section. We show that ${}^A\mathcal{P}(Q)$ acts on the equivariant A -homology of Nakajima quiver varieties.

5.1. Preliminaries on Nakajima quiver variety. In this subsection, we recall the definition of Nakajima quiver varieties in [Nak94].

For a quiver Q , we introduce the *framed quiver* Q^\heartsuit , whose set of vertices is $I \sqcup I'$, where I' is another copy of the set I , equipped with the bijection $I \rightarrow I'$, $i \mapsto i'$. The set of arrows of Q^\heartsuit is, by definition, the disjoint union of H and a set of additional edges $j_i : i \rightarrow i'$, one for each vertex $i \in I$. We follow the tradition that $v \in \mathbb{N}^I$ is the notation for the dimension vector at I , and $w \in \mathbb{N}^{I'}$ is the dimension vector at I' . We denote $\text{Rep}(Q^\heartsuit, (v, w))$ simply by $\text{Rep}(Q, v, w)$.

Let $\overline{Q^\heartsuit} = Q^\heartsuit \sqcup Q^{\heartsuit, \text{op}}$ be the double of Q^\heartsuit . We have the isomorphism

$$\text{Rep}(\overline{Q^\heartsuit}, (v, w)) = T^* \text{Rep}(Q, v, w) = \text{Rep}(Q, v) \times \text{Rep}(Q^{\text{op}}, v) \times \text{Hom}_{kI}(W, V) \times \text{Hom}_{kI}(V, W).$$

Let $\mu_{v,w} : T^* \text{Rep}(Q, v, w) \rightarrow \mathfrak{gl}_v^* \cong \mathfrak{gl}_v$ be the moment map

$$\mu_{v,w} : (x, x^*, i, j) \mapsto \sum [x, x^*] + i \circ j \in \mathfrak{gl}_v.$$

For any $\theta = (\theta_i)_{i \in I} \in \mathbb{Z}^I$, let $\chi_\theta : G_v \rightarrow \mathbb{G}_m$ be the character $g = (g_i)_{i \in I} \mapsto \prod_{i \in I} \det(g_i)^{-\theta_i}$. The set of χ_θ -semistable points in $T^* \text{Rep}(Q, v, w)$ is denoted by $\text{Rep}(\overline{Q}, v, w)^{\text{ss}}$. The Nakajima quiver variety is defined to be the Hamiltonian reduction

$$\mathfrak{M}_\theta(v, w) := \mu_{v,w}^{-1}(0) //_\theta G_v.$$

The following description of stability condition can be found in [Gin09, Corollary 5.1.9]. When $\theta = \theta^+ = (1, \dots, 1)$, the point $(x, x^*, i, j) \in \mu^{-1}(0)$ is θ^+ -semistable, if and only if, the following holds: For any collection of vector subspaces $S = (S_i)_{i \in k} \subset V = (V_i)_{i \in k}$, which is stable under the maps x and x^* , if $S_k \subset \ker(j_k)$ for any $k \in I$, then $S = 0$.

In this paper, we use the stability condition θ^+ unless otherwise specified.

5.2. Representations from quiver varieties. Let $v_1, v_2 \in \mathbb{N}^I$ be two dimension vectors. Let $v = v_1 + v_2$. We fix an I -tuple of vector spaces V of dimension vector v . We fix $V_1 \subset V$ an I -tuple of subspaces of V with dimension vector v_1 . Let $V_2 := V/V_1$, with the projection map $\text{pr}_2 : V \rightarrow V_2$. As in Section §1.4, we set $G = G_v$, and $P = \{g \in G \mid g(V_1) = V_1\}$. We consider the Lagrangian correspondence formalism in §1.4, specialized as follows: We take X' to be $\text{Rep}(Q, v, w)$ and Y to be $\text{Rep}(Q, v_1, w) \times \text{Rep}(Q, v_2)$. Define \mathcal{V} to be

$$\mathcal{V} := \{(x, j) \in \text{Rep}(Q, v_1 + v_2, w) \mid x(V_1) \subset V_1\} \subset X'.$$

As in §1.4, set $X := G \times_P Y$, $W := G \times_P \mathcal{V}$, and $Z := T_W^*(X \times X')$ the conormal bundle of W . We then have the correspondence

$$T^*X \xleftarrow{\phi} Z \xrightarrow{\psi} T^*X'.$$

Lemma 5.1. *Notations are as above.*

(a) *We have the following canonical isomorphisms of G -varieties.*

$$T^*X' = T^*\text{Rep}(Q, v_1 + v_2, w) = \text{Rep}(\overline{Q}, v_1 + v_2, w).$$

$$T^*X = G \times_P \{(c, x, x^*, i, j) \mid c \in \mathfrak{p}_v, x \in \text{Rep}(Q, v_1) \times \text{Rep}(Q, v_2), x^* \in \text{Rep}(Q^{op}, v_1) \times \text{Rep}(Q^{op}, v_2), \\ j \in \text{Hom}(V_1, W), i \in \text{Hom}(W, V_1), [x, x^*] + i \circ j = \text{pr}(c)\},$$

where $\text{pr}(c)$ is the projection of c in $\mathfrak{g}_{v_1} \oplus \mathfrak{g}_{v_2}$.

$$Z = G \times_P \{(x, x^*, i, j) \in T^*\text{Rep}(Q, v_1 + v_2, w) \mid (x, x^*)(V_1) \subset V_1, \text{Im}(i) \subset V_1\}.$$

(b) *For $(g, x, x^*, i, j) \in Z$, the maps ϕ, ψ are given by*

$$\begin{aligned} \phi((g, x, x^*, i, j) \bmod P) &= (g, [x, x^*] + i \circ j, \text{pr}(x), \text{pr}(x^*), i^{V_1}, j_{V_1}) \bmod P, \\ \psi((g, x, x^*, i, j) \bmod P) &= (gxg^{-1}, gx^*g^{-1}, jg^{-1}, gi). \end{aligned}$$

(c) *We have the following canonical isomorphisms of G -varieties.*

$$T_G^*X' = \mu_{v,w}^{-1}(0).$$

$$T_G^*X = G \times_P (\mu_{v_1,w}^{-1}(0) \times \mu_{v_2}^{-1}(0)).$$

$$Z_G = G \times_P \{(x, x^*, i, j) \in \mu_{v,w}^{-1}(0) \mid (x, x^*)(V_1) \subset V_1, \text{Im}(i) \subset V_1\}.$$

The maps $\overline{\phi} : Z_G \rightarrow T_G^*X$ and $\overline{\psi} : Z_G \rightarrow T_G^*X'$ are induced from ϕ, ψ in (b).

Proof. The proof goes the same way as [SV12, Lemma 7.4]. We only explain how to get the formula of T^*X in (a) here. The rest are similar. By Lemma 7.1 of [SV12], we have $T^*X = T_P^*(G \times Y)/P$. Thus,

$$(6) \quad \begin{aligned} T^*X &= G \times_P \{(f, a) \in (\mathfrak{g} \times \text{Rep}(Q, v_1, w) \times \text{Rep}(Q, v_2))^* \times (\text{Rep}(Q, v_1, w) \times \text{Rep}(Q, v_2)) \\ &\quad \mid f(-b, [\text{pr}(b), a]) = 0, \forall b \in \mathfrak{p}\}. \end{aligned}$$

For (f, a) as in (6), we write $f = f_1 \times f_2$, where

$$f_1 \in \mathfrak{g}^*, \text{ and } f_2 \in (\text{Rep}(Q, v_1, w) \times \text{Rep}(Q, v_2))^*.$$

Write $\mathfrak{h} := \mathfrak{gl}_{v_1} \times \mathfrak{gl}_{v_2}$ for short. Starting with an element (f, a) in (6), we define an element $\tilde{f} \in (\mathfrak{g} \times \mathfrak{h})^*$ by $\tilde{f}(g, h) := f(g, [h, a])$. Let $\delta' : \mathfrak{p} \rightarrow \mathfrak{g} \times \mathfrak{h}$ be the linear function $b \mapsto (b, -\text{pr}(b))$. Therefore, we have $\tilde{f}(\delta'(\mathfrak{p})) = 0$.

Identify $(\mathfrak{g} \times \mathfrak{h})^*$ with $\mathfrak{g} \times \mathfrak{h}$ via $(g_1, g_2) := \text{tr}(g_1 g_2)$. Let $\delta : \mathfrak{p} \rightarrow \mathfrak{g} \times \mathfrak{h}$ be the linear function $b \mapsto (b, \text{pr}(b))$. Then, $\delta' \mathfrak{p}^\perp$ is identified with $\delta \mathfrak{p} \cong \mathfrak{p}$. Therefore, $\tilde{f} \in (\mathfrak{g} \times \mathfrak{h})^*$ corresponds to

$(c, \text{pr}(c)) \in \delta \mathfrak{p}$ for some $c \in \mathfrak{p}$, and its first component f_1 corresponds to c under the identification $\mathfrak{g}^* \cong \mathfrak{g}$. The second component f_2 of f satisfies

$$(7) \quad f_2([h, a]) = \text{tr}(\text{pr}(c) \cdot h), \text{ for any } h \in \mathfrak{h}.$$

For vector spaces E, F , the bilinear function

$$\text{Hom}(E, F) \times \text{Hom}(F, E) \rightarrow k, \quad (f, f^*) \mapsto \text{tr}(f \circ f^*).$$

gives an isomorphism $\text{Hom}(E, F)^* \cong \text{Hom}(F, E)$. We identify the second component f_2 with some element $b \in \text{Rep}(Q^{op}, v_1, w) \times \text{Rep}(Q^{op}, v_2)$. The equality (7) yields $[a, b] = \text{pr}(c)$. This proves (a). \square

We will abbreviate $T^*Y^s = \text{Rep}(\overline{Q}, v_1, w)^{ss} \times \text{Rep}(\overline{Q}, v_2) \subseteq T^*Y$, and $T^*X'^s = \text{Rep}(\overline{Q}, v, w)^{ss} \subseteq T^*X'$. There is a bundle projection $T^*X \rightarrow G \times_P T^*Y$. We define T^*X^s to be the preimage of $G \times_P T^*Y^s$ under this bundle projection. In particular, we have

$$T_G^*X^s := G \times_P (T_L^*Y \cap T^*Y^s) = G \times_P (\mu_{v_1, w}^{-1}(0)^{ss} \times \mu_{v_2}^{-1}(0)),$$

for $L = G_{v_1} \times G_{v_2}$. We define $Z^s := \psi^{-1}(T^*X'^s)$ and $Z_G^s := Z^s \cap Z_G$.

Lemma 5.2. *We have $\phi(Z^s) \subset T^*X^s$.*

Proof. This follows from the description of stability condition θ^+ in § 5.1. \square

Thus, we have the following diagram of correspondences:

$$(8) \quad \begin{array}{ccccc} T_G^*X^s & \xleftarrow{\overline{\phi}} & Z_G^s & \xrightarrow{\overline{\psi}} & T_G^*X' \cap T^*X'^s \\ \downarrow & & \downarrow & & \downarrow \\ G \times_P T^*Y^s & \xrightarrow{\iota} & T^*X^s & \xleftarrow{\phi} & Z^s & \xrightarrow{\psi} & T^*X'^s. \end{array}$$

The diagram (8) is a diagram of $G_v \times T \times G_w$ -varieties. By Lemma 5.1, we have:

$$\begin{aligned} Z_G^s &= G \times_P \{(x, x^*, i, j) \in \mu_{v, w}^{-1}(0)^{ss} \mid (x, x^*)(V_1) \subset V_1, \text{Im}(i) \subset V_1\}. \\ T_G^*X^s &= G \times_P (\mu_{v_1, w}^{-1}(0)^{ss} \times \mu_{v_2}^{-1}(0)). \end{aligned}$$

The left square of diagram (8) is a pullback diagram.

Remark 5.3. Lemma 5.2 will fail if θ is not in the same chamber as θ^+ .

Let $v, w \in \mathbb{N}^I$ be the dimension vectors. As the action of G_v on $\mu_{v, w}^{-1}(0)^{ss} := \mu_{v, w}^{-1}(0) \cap \text{Rep}(\overline{Q}, v, w)^{ss}$ is free, we have

$$\mathcal{M}(v, w) := A_{T \times G_w}(\mathfrak{M}(v, w)) \cong A_{G_v \times T \times G_w}(\mu_{v, w}^{-1}(0)^{ss}).$$

For each $w \in \mathbb{N}^I$, and each pair $v_1, v_2 \in \mathbb{N}^I$, we define maps

$$a_{v_1, v_2} : \mathcal{M}(v_1, w) \otimes \mathcal{P}_{v_2} \rightarrow \mathcal{M}(v_1 + v_2, w)$$

as follows.

We start with the Künneth morphism.

$$(9) \quad \begin{aligned} \mathcal{M}(v_1, w) \times \mathcal{P}_{v_2} &= A_{G_{v_1} \times G_w \times T}(\mu_{v_1, w}^{-1}(0)^{ss}) \otimes A_{G_{v_2} \times T}(\mu_{v_2}^{-1}(0)) \\ &\rightarrow A_{G_{v_1} \times G_{v_2} \times T \times G_w}(\mu_{v_1, w}^{-1}(0)^{ss} \times \mu_{v_2}^{-1}(0)) \\ &\cong A_{G \times T \times G_w}(G \times_P (\mu_{v_1, w}^{-1}(0)^{ss} \times \mu_{v_2}^{-1}(0))). \end{aligned}$$

We define the map a_{v_1, v_2} to be the composition of the morphism in (9) with the following morphism

$$\overline{\psi}_* \circ \phi^\sharp : A_{G_v \times T \times G_w} (T_G^* X^s) \rightarrow A_{G_v \times T \times G_w} (T_G^* X' \cap T^* X'^s) = \mathcal{M}(v, w),$$

where the pullback ϕ^\sharp is the Gysin pullback of ϕ .

Theorem 5.4. *For each $w \in \mathbb{N}^I$, the maps*

$$a_{v_1, v_2} : \mathcal{M}(v_1, w) \otimes \mathcal{P}_{v_2} \rightarrow \mathcal{M}(v_1 + v_2, w)$$

fit together to define an action of ${}^A\mathcal{P}$ on $\mathcal{M}(w) := \bigoplus_v \mathcal{M}(v, w)$. In other words, a_{v_1, v_2} induces a $R[[t_1, t_2]]$ -algebra homomorphism

$$\Phi : {}^A\mathcal{P} \rightarrow \text{End}\left(\bigoplus_{v \in \mathbb{N}^I} A_{T \times G_w}(\mathfrak{M}(v, w))\right).$$

Proof. The proof follows from the same idea as the proof of Theorem 4.1. More precisely, we fix a flag $V_1 \subset V_2 \subset V$, with $\dim V_i = v_1 + \dots + v_i$. Fix the vector space W with dimension vector w . We define the following varieties:

- Let X_1 be the set of quadruples (F_1, F_2, a, j) , where $F_1 \subset F_2 \subset V$ is a flag such that $F_1 \cong V_1, F_2 \cong V_2$, and $a \in \text{Rep}(Q, v)$ is an endomorphism of the vector space $F_1 \oplus (F_2/F_1) \oplus (V/F_2)$. j is an element of $\text{Hom}(F_1, W)$.
- Let X_2 be the set of pairs (F_1, a, j) , where $F_1 \subset V$, such that $F_1 \cong V_1$ and $a \in \text{Rep}(Q, v)$ is an endomorphism of the vector space $F_1 \oplus (V/F_1)$. j is an element of $\text{Hom}(F_1, W)$.
- $X_3 = \text{Rep}(Q, v) \oplus \text{Hom}(V, W)$.

We then define the following correspondences. For $i = 1, 2, 3$, let W_i be the following.

$$W_1 = \{(F_1, a, j) \mid F_1 \subset V, \text{ such that } F_1 \cong V_1, \text{ and } a(F_1) \subset F_1, \text{ for } a \in \text{Rep}(Q, v), j \in \text{Hom}(V, W)\}.$$

$$W_2 = \{(F_1, F_2, a, j) \mid F_1 \subset F_2 \subset V, a \in \text{Rep}(Q, v), \text{ such that } F_i \cong V_i, \text{ and } a(F_i) \subset F_i, \\ \text{for } i = 1, 2, \text{ and } j \in \text{Hom}(V, W)\}.$$

$$W_3 = \{(F_1, F_2, a, v) \mid F_1 \subset F_2 \subset V, a \in \text{Rep}(Q, v), \text{ such that } F_i \cong V_i, \text{ for } i = 1, 2, \text{ and} \\ a \in \text{End}(F_1 \oplus (V/F_1)), a \text{ preserves the subspace } \{0\} \oplus (F_2/F_1), \text{ and } j \in \text{Hom}(F_1, W)\}.$$

We have the inclusions $W_1 \subset X_3 \times X_2$, $W_2 \subset X_3 \times X_1$, and $W_3 \subset X_2 \times X_1$. It is clear that those inclusions give an isomorphism $W_2 = W_1 \times_{X_2} W_3$. We consider

$$Z_1 = T_{W_1}^*(X_3 \times X_2), \quad Z_2 = T_{W_2}^*(X_3 \times X_1), \quad Z_3 = T_{W_3}^*(X_2 \times X_1).$$

The intersection $(W_1 \times X_1) \cap (X_3 \times W_2)$ is transversal in $X_3 \times X_2 \times X_1$. Thus, by [CG, Theorem 2.7.26] we have an isomorphism $Z_1 \times_{T^*X_2} Z_3 \cong Z_2$. As in Theorem 4.1, we have $\dim(Z_1) + \dim(Z_3) = \dim(Z_2) + \dim(T^*X_2)$ by Lemma 1.16.

Let $P_1 := \{g \in G = G_{v_1+v_2+v_3} \mid g(V_1) \subset V_1\}$, with Lie algebra \mathfrak{p}_1 , and $P := \{g \in G \mid g(V_i) \subset V_i, i = 1, 2\}$ with Lie algebra \mathfrak{p} . By Lemma 5.1, we have

$$T^*X_2 \subset G \times_{P_1} (\mathfrak{p}_1 \times \text{Rep}(\overline{Q^\vee}, v_1, w) \times \text{Rep}(\overline{Q}, v_2 + v_3)), \\ T^*X_1 \subset G \times_P (\mathfrak{p} \times \text{Rep}(\overline{Q^\vee}, v_1, w) \times \text{Rep}(\overline{Q}, v_2) \times \text{Rep}(\overline{Q}, v_3)).$$

Define

$$T^*X_3^s := \text{Rep}(\overline{Q^\vee}, v_1 + v_2 + v_3, w)^{ss}, \\ T^*X_2^s := T^*X_2 \cap G \times_{P_1} (\mathfrak{p}_1 \times \text{Rep}(\overline{Q^\vee}, v_1, w)^{ss} \times \text{Rep}(\overline{Q}, v_2 + v_3)), \\ T^*X_1^s := T^*X_1 \cap G \times_P (\mathfrak{p} \times \text{Rep}(\overline{Q^\vee}, v_1, w)^{ss} \times \text{Rep}(\overline{Q}, v_2) \times \text{Rep}(\overline{Q}, v_3)).$$

Define

$$Z_3^s := \psi_3^{-1}(T^*X_2^s), \quad Z_2^s := \psi_2^{-1}(T^*X_3^s), \quad Z_1^s := \psi_1^{-1}(T^*X_3^s).$$

Then we have the following diagram with the square being Cartesian.

$$\begin{array}{ccccc} T^*X_3^s & \xleftarrow{\psi_1} & Z_1^s & \xrightarrow{\phi_1} & T^*X_2^s, \\ & \searrow \psi_2 & \uparrow & & \uparrow \psi_3 \\ & & Z_2^s & \xrightarrow{\quad} & Z_3^s, \\ & & \searrow \phi_2 & & \downarrow \phi_3 \\ & & & & T^*X_1^s. \end{array}$$

We define the maps I_1, I_2, I_3 as in Theorem 4.1. The same argument shows $I_2 = I_1 \circ I_3$. This implies a_{v_1, v_2} is an action map. \square

By convention, we consider the right action of \mathcal{P} on $\mathcal{M}(w)$. Therefore, the multiplication of $\text{End}(\mathcal{M}(w))$ is understood as composition of right action operators.

Remark 5.5. If one uses the stability condition θ^- in the definition of Nakajima quiver variety, the Lagrangian correspondence $Z_G^{\chi_{\theta^-} - ss}$ should be adjusted to

$$Z_G^{\chi_{\theta^-} - ss} = G \times_P \{(x, x^*, i, j) \in \mu_{v,w}^{-1}(0)^{\chi_{\theta^-} - ss} \mid (x, x^*)(V_1) \subset V_1, \ker(j) \supset V_1\}.$$

The Lagrangian correspondence formalism will give us a left action of \mathcal{P}^{op} on $A_{G_w \times T}(\mathfrak{M}_{\theta^-}(w))$. This left module of \mathcal{P}^{op} coincides with the natural left action of \mathcal{P}^{op} on $\mathcal{M}(w)$, under the identification of $\mathfrak{M}_{\theta^+}(v, w)$ and $\mathfrak{M}_{\theta^-}(v, w)$, sending any representation V to its dual V^\vee .

5.3. Nakajima's raising operators. In this section, we interpret the action of ${}^A\mathcal{P}(Q)^s$ constructed in the previous subsection in terms of Nakajima's raising operators. This interpretation allows us to compare ${}^A\mathcal{P}(Q)^s$ with the quantum groups.

We start by recalling the raising operators constructed by Nakajima in [Nak98, Nak01]. Recall in Section §5.1, we denote by $\mathfrak{M}(v, w)$ the Nakajima quiver variety with the fixed stability condition θ^+ . Let $\mathfrak{M}_0(v, w)$ be the affine quotient of $\mu_{v,w}^{-1}(0)$. That is,

$$\mathfrak{M}_0(v, w) := \text{Spec}(k[\mu_{v,w}^{-1}(0)]^{G_v}),$$

where $k[\mu_{v,w}^{-1}(0)]$ is the coordinate ring of $\mu_{v,w}^{-1}(0)$. We have the resolution of singularities: $\pi : \mathfrak{M}(v, w) \rightarrow \mathfrak{M}_0(v, w)$. For two dimension vectors v_1 and v_2 , the composition $\mathfrak{M}(v_i, w) \rightarrow \mathfrak{M}_0(v_i, w) \subset \mathfrak{M}_0(v_1 + v_2, w)$ are still denoted by π_i . Let

$$Z(v_1, v_2, w) := \{(x_1, x_2) \in \mathfrak{M}(v_1, w) \times \mathfrak{M}(v_2, w) \mid \pi_1(x_1) = \pi_2(x_2)\}$$

be the Steinberg variety. By the construction of $\mathfrak{M}(v, w)$, we have the tautological vector bundle

$$\mu_v^{-1}(0)^{ss} \times_{G_v} V \rightarrow \mathfrak{M}(v, w)$$

associated to the principal G_v -bundle $\mu_v^{-1}(0)^{ss} \rightarrow \mathfrak{M}(v, w)$. Here V is the G_v representation with dimension vector v . We denote the vector bundle by $\mathcal{V}(v, w)$.

In the special case when $v_1 = v_2 - e_k$, where e_k is the dimension vector whose entry k is 1, and other entries are 0. The Hecke correspondence $C_k^+(v_2, w)$ (see [Nak98, Nak01]) is an irreducible component of $Z(v_1, v_2, w)$, defined as the set of quintuples $\{(x, x^*, i, j, S)\}$ up to G_v -conjugation, where $(x, x^*, i, j) \in \mu_{v,w}^{-1}(0)^{ss}$ and $S \subset V$ is a x, x^* -invariant subspace containing the image of i with $\dim(S) = v_2 - e_k$. We consider $C_k^+(v_2, w)$ as a closed subvariety of $\mathfrak{M}(v_2 - e_k, w) \times \mathfrak{M}(v_2, w)$ by setting

$$(B^1, i^1, j^1) := \text{the restriction of } (B, i, j) \text{ to } S, \quad (B^2, i^2, j^2) := (B, i, j).$$

The component $C_k^+(v_2, w)$ is smooth, and it is a Lagrangian subvariety of $\mathfrak{M}(v_2 - e_k, w) \times \mathfrak{M}(v_2, w)$ as shown by Nakajima. In particular,

$$\dim C_k^+(v_2, w) = \frac{\dim \mathfrak{M}(v_2 - e_k, w) + \dim \mathfrak{M}(v_2, w)}{2}.$$

The tautological line bundle \mathcal{L}_k of $C_k^+(v_2, w)$ is defined to be the quotient

$$\mathcal{L}_k := \mathcal{V}(v_2, w) / \mathcal{V}(v_1, w).$$

Nakajima defined the following raising operators. Let $f(t) \in A_T(\text{pt})[[t]]$ be a power series. Then $f(c_1(\mathcal{L}_k))$ is a well-defined element in $A_{G_w \times T}(C_k^+(v_2, w))$. We have the following diagram:

$$\begin{array}{ccc} C_k^+(v_2, w) & \hookrightarrow & \mathfrak{M}(v_2 - e_k, w) \times \mathfrak{M}(v_2, w) \\ & \swarrow & \searrow \\ \mathfrak{M}(v_2 - e_k, w) & & \mathfrak{M}(v_2, w). \end{array}$$

Denote by $p_i : C_k^+(v_2, w) \rightarrow \mathfrak{M}(v^i, w)$ the composition of the inclusion with the i -th projections, for $i = 1, 2$, and $v_1 = v_2 - e_k$. Let $\Psi(f(c_1(\mathcal{L}_k))) \in \text{End}_{R[[t_1, t_2]]}(A_{G_w \times T}(\mathfrak{M}(w)))$ be the raising operation given by convolution with $f(c_1(\mathcal{L}_k))$. In other words, let $\alpha \in A_{G_w \times T}(\mathfrak{M}(v_1, w))$,

$$\Psi(f(c_1(\mathcal{L}_k)))(\alpha) := p_{2*}(p_1^*(\alpha) \cap f(c_1(\mathcal{L}_k))).$$

For the dimension vector e_k , the condition

$$\mu_{e_k} : \text{Rep}(Q, e_k) \times \text{Rep}(Q^{\text{op}}, e_k) \rightarrow k, \quad [x, x^*] = 0$$

means that for each edge-loop, the pair x, x^* form two free polynomial variables. Therefore $\mu_{e_k}^{-1}(0)$ is a vector space with $G_{e_k} = \mathbb{G}_m$ -action. Then, we have

$$\mathcal{P}_{e_k} := A_{G_v \times T}(\mu_{e_k}^{-1}(0)) \cong A_{\mathbb{G}_m \times T}(\text{pt}) \cong A_T(\text{pt})[[z^{(k)}]].$$

Let ξ_k be the natural one dimensional representation of $G_{e_k} = \mathbb{G}_m$. Then, $z^{(k)}$ can be viewed as $c_1(\xi_k) \in A_{G_v \times T}(\mu_{e_k}^{-1}(0))$.

In the case of $v_1 + e_k = v_2$, we write $v = v_2$ for short, for the Lagrangian correspondence, we have $Y = \text{Rep}(Q, v - e_k, w) \times \text{pt}$, $X' = \text{Rep}(Q, v, w)$. Therefore, the correspondence in this case becomes

$$(10) \quad G_v \times_P \mu_{v-e_k, w}^{-1}(0)^{ss} \xleftarrow{\bar{\phi}} Z_G^s \xrightarrow{\bar{\psi}} \mu_{v, w}^{-1}(0)^{ss}.$$

Theorem 5.6. *For any $f(t) \in A_T(\text{pt})[[t]]$, view $f(z^{(k)}) \in \mathcal{P}_{e_k} \cong A_T(\text{pt})[[z^{(k)}]]$, we have the equality*

$$\Psi(f(c_1(\mathcal{L}_k))) = \Phi(f(z^{(k)}))$$

in $\text{End}_{R[[t_1, t_2]]}(A_{G_w \times T}(\mathfrak{M}(w)))$, where Φ is the action of the preprojective CoHA ${}^A\mathcal{P}$.

Proof. Taking the quotient by G_v of the Lagrangian correspondence (10), we get the following commutative diagram.

$$\begin{array}{ccccc} T^*X^s & \xleftarrow{\phi} & Z^s & \xrightarrow{\psi} & T^*X'^s \\ \uparrow g & & \uparrow g' & & \uparrow \\ G_v \times_P \mu_{v-e_k}^{-1}(0)^{ss} & \xleftarrow{\bar{\phi}} & Z_G^s & \xrightarrow{\bar{\psi}} & \mu_v^{-1}(0)^{ss} \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{M}(v - e_k, w) & \xleftarrow{p_1} & C_k^+(v, w) & \xrightarrow{p_2} & \mathfrak{M}(v, w) \end{array}$$

The vertical maps g, g' are closed embeddings. Here recall that

$$G_v \times_P \mu_{v-e_k}^{-1}(0)^{ss}/G_v \cong \mu_{v-e_k}^{-1}(0)^{ss}/G_{v-e_k} = \mathfrak{M}(v-e_k, w).$$

In the above diagram, the map $\bar{\phi}$ is a smooth morphism of smooth varieties. The usual pullback $\bar{\phi}^*$ is well-defined. We first show the Gysin pullback ϕ^\sharp is the same as the usual pullback $\bar{\phi}^*$. By Lemma 5.1, the variety Z_G^s is a principal G_v -bundle of $C_k^+(v, w)$. Thus,

$$\begin{aligned} \dim Z_G^s &= \dim G_v + \dim C_k^+(v, w) = \dim G_v + \frac{\dim \mathfrak{M}(v-e_k, w) + \dim \mathfrak{M}(v, w)}{2} \\ &= \dim G_v + \frac{2 \dim \operatorname{Rep}(Q, v-e_k, w) - 2 \dim G_{v-e_k} + 2 \dim \operatorname{Rep}(Q, v, w) - 2 \dim G_v}{2} \\ &= \dim \operatorname{Rep}(Q, v-e_k, w) - \dim G_{v-e_k} + \dim \operatorname{Rep}(Q, v, w) \\ &= w \cdot (v-e_k) + A_Q(v-e_k) \cdot (v-e_k) - (v-e_k) \cdot (v-e_k) + w \cdot v + A_Q v \cdot v; \\ \dim T^*X &= 2 \dim(G \times_P Y) = 2(\dim(G_v/P) + \dim \operatorname{Rep}(Q, v-e_k, w) + \dim \operatorname{Rep}(Q, e_k)) \\ &= 2(\dim(G_v/P) + \dim \operatorname{Rep}(Q, v-e_k, w)); \\ \dim(G \times_P \mu_{v-e_k, w}^{-1}(0)^{ss}) &= \dim(G_v/P) + \dim \mu_{v-e_k, w}^{-1}(0)^{ss} \\ &= \dim(G_v/P) + 2 \dim \operatorname{Rep}(Q, v-e_k, w) - \dim G_{v-e_k}; \\ \dim Z &= \dim G_v/P + 2(\operatorname{Rep}(Q, v-e_k) + A_Q e_k \cdot v) + w \cdot (v-e_k) + w \cdot v \\ &= \dim G_v/P + 2(A_Q(v-e_k) \cdot (v-e_k) + A_Q e_k \cdot v) + w \cdot (v-e_k) + w \cdot v. \end{aligned}$$

Therefore,

$$\dim Z - \dim Z_G = \dim(G_v/P) + \dim G_{v-e_k} = \dim T^*X - \dim(G \times_P \mu_{v-e_k, w}^{-1}(0)^{ss}).$$

Hence we have

$$\dim(T^*X^s) + \dim(Z_G^s) = \dim(G \times_P \mu_{v-e_k, w}^{-1}(0)^{ss}) + \dim Z^s.$$

Thus, Lemma 1.13(2) yields $\phi^* \circ g_* = g'_* \circ \bar{\phi}^*$. Therefore, for $\alpha \in A_{G_w \times T}(\mathfrak{M}(v-e_k, w))$,

$$\Phi((z^{(k)})^l)(\alpha) = \bar{\psi}_* \bar{\phi}^*((z^{(k)})^l \otimes \alpha),$$

here $\bar{\phi}^*$ is the usual pullback. Here to distinguish the vertex $k \in I$ and the power $l \in \mathbb{N}$, we write (k) for the label $k \in I$.

The isomorphism $Z_G^s/G_v \cong C_k^+(v, w)$ follows from Lemma 5.1. It induces an isomorphism

$$A_{G \times T \times G_w}(Z^s \cap Z_G) \cong A_{T \times G_w}(C_k^+(v, w)).$$

The isomorphism maps $\bar{\phi}^*((z^{(k)})^l \otimes \alpha)$ to $(c_1(\mathcal{L}_k))^l \otimes p_1^*(\alpha)$, for any l . The pullback of the line bundle \mathcal{L}_k on Z_G^s is the trivial bundle with fiber $V(v, w)/V(v-e_k, w)$. It carries a natural $G_{e_k} = \mathbb{G}_m$ action. The element $z^{(k)}$ can be interpreted as $z^{(k)} = c_1(V(v, w)/V(v-e_k, w)) \in A_{G_v \times T}(\mu_{e_k}^{-1}(0))$. Thus, $\bar{\phi}^*(z^{(k)}) \mapsto c_1(\mathcal{L}_k)$ under the isomorphism.

The claim follows now from the definitions of the two actions Ψ and Φ . \square

5.4. Extended CoHA. For later use, we will consider some modifications of \mathcal{P} .

Assumption 5.7. Assume the action of $T = \mathbb{G}_m^2$ on $T^* \operatorname{Rep}(Q, v)$ satisfies one of the following:

Case 1: $m(h) = m(h^*) = 1$, for any $h \in H$;

Case 2: The T -action is the one defined in Remark 3.2.

Note that Assumption 3.1 is satisfied in either case above.

Let (R, F) be the formal group law of A . Define ${}^A\mathcal{P}^0 := \text{Sym}_R(\oplus_{i \in I} A_{G_{e_i}}(\text{pt}))$ be the symmetric algebra of $\oplus_{i \in I} A_{G_{e_i}}(\text{pt})$. Let

$$\Phi_k(z) = \sum_{r \geq 0} (u^{(k)})^r z^{-r} \in {}^A\mathcal{P}^0[[z^{-1}]]$$

be the generating series of generators $(u^{(k)})^r \in {}^A\mathcal{P}^0$.

Recall that a_{ik} is the number of arrows in H from i to k . Let $c_{ik} = -a_{ik} - a_{ki}$ if $k \neq i$, 2 if $k = i$. We define a ${}^A\mathcal{P}^0$ action on ${}^A\mathcal{P}$ as follows. For any $g \in {}^A\mathcal{P}_v$,

$$\Phi_k(z)g\Phi_k(z)^{-1} := g\widehat{\Phi}_k(z, v),$$

where

$$\widehat{\Phi}_k(z, v) := \prod_{i \in I \setminus \{k\}} \prod_{j=1}^{v^i} \frac{(z - {}_F\lambda_j^{(i)} - {}_F t_1)^{a_{ik}} (z - {}_F\lambda_j^{(i)} - {}_F t_2)^{a_{ki}}}{(z - {}_F\lambda_j^{(i)} + {}_F t_2)^{a_{ik}} (z - {}_F\lambda_j^{(i)} + {}_F t_1)^{a_{ki}}} \cdot \prod_{j=1}^{v^k} \frac{(z - {}_F\lambda_j^{(k)} + {}_F t_1 + {}_F t_2)}{(z - {}_F\lambda_j^{(k)} - {}_F t_1 - {}_F t_2)}$$

when the T -action satisfies Assumption 5.7(1); and

$$\widehat{\Phi}_k(z, v) := \prod_{i \in I} \prod_{j=1}^{v^i} \frac{(z - {}_F\lambda_j^{(i)} + {}_F(c_{ki})_F \frac{\hbar}{2})}{(z - {}_F\lambda_j^{(i)} - {}_F(c_{ki})_F \frac{\hbar}{2})}$$

when the T -action satisfies Assumption 5.7(2). Note that the element $\widehat{\Phi}_k(z, v)$ lies in $A_{\text{GL}_v \times T}(\text{pt})[[z^{-1}]]$ in both cases.

Lemma 5.8. (1) The action of ${}^A\mathcal{P}^0$ on ${}^A\mathcal{P}$ is well-defined.

(2) Furthermore, for any $k \in I$, Φ_k acts on \mathcal{P} by an algebra homomorphism.

(3) The subalgebra $\mathcal{P}^s \subseteq \mathcal{P}$ is a \mathcal{P}^0 -submodule.

Proof. (1) follows from the fact that $\widehat{\Phi}_i(z, v)$ is symmetric. (2) follows from the projection formula and the equality $\widehat{\Phi}_i(z, v_1) \cdot \widehat{\Phi}_i(z, v_2) = \widehat{\Phi}_i(z, v_1 + v_2)$. (3) is clear. \square

Definition 5.9. Define the extended preprojective CoHA associated to A and Q to be the algebra ${}^A\mathcal{P}^\epsilon = {}^A\mathcal{P}^0 \ltimes {}^A\mathcal{P}$. The spherical subalgebra in the extended CoHA is ${}^A\mathcal{P}^{s, \epsilon} = {}^A\mathcal{P}^0 \ltimes {}^A\mathcal{P}^s$.

Similarly, there is an action of ${}^A\mathcal{P}^0$ on ${}^A\mathcal{SH}$ by algebra homomorphism. We define the extension of shuffle algebra ${}^A\mathcal{SH}^\epsilon := {}^A\mathcal{P}^0 \ltimes {}^A\mathcal{SH}$. As a direct consequence of Proposition 4.3, we have the following.

Proposition 5.10. There is an algebra homomorphism ${}^A\mathcal{P}^\epsilon \rightarrow \mathcal{SH}^\epsilon$, which becomes an isomorphism after localization.

5.5. Action on quiver varieties. For any G -variety X and a G -equivariant rank- n vector bundle V , let $\lambda_{-1/z}(V) \in A_G(X)[[z]]$ be the equivariant Chern polynomial $\prod_{i=1}^n (z - {}_F x_i)$ where $\{x_1, \dots, x_n\}$ are the Chern roots of V . Note that for any element ξ in $K_G(X)$, the Grothendieck ring of G -equivariant vector bundles on X , $\lambda_{-1/z}(\xi)$ is well-defined.

Let $\mathfrak{M}(v, w) := \mu_{v, w}^{-1}(0)^{ss}/G_v$ be the quiver variety, and let $\mathcal{V}_k := \mu_{v, w}^{-1}(0)^{ss} \times_{G_v} V_k$ (resp. \mathcal{W}_k) be the k -th tautological bundle (resp. the trivial W_k -bundle) on $\mathfrak{M}(v, w)$ respectively. Consider the following tautological element in $K^{G_w \times T}(\mathfrak{M}(v, w))$.

$$\mathcal{F}_k(v, w) = \begin{cases} q_1^{-1} \mathcal{W}_k - (1 + (q_1 q_2)^{-1} \mathcal{V}_k) & + q_1^{-1} \sum_{\{h \in H | \text{in}(h)=k\}} V_{\text{out}(h)} \\ & + q_2^{-1} \sum_{\{h \in H | \text{out}(h)=k\}} V_{\text{in}(h)} \\ q^{-1} \mathcal{W}_k - (1 + q^{-2}) \mathcal{V}_k + q^{-1} \sum_{l: k \neq l} [-c_{kl}]_q \mathcal{V}_l & \end{cases} \quad \begin{array}{l} \text{under Assumption 5.7(1);} \\ \text{under Assumption 5.7(2).} \end{array}$$

where $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. This is a modification of [Nak01, (2.9.1)] according to T -action. When $t_1 \neq t_2$, the T action is given by $(t_1, t_2)(B_h, B_{h^*}, i, j) = (t_1 B_h, t_2 B_{h^*}, t_1 i, t_2 j)$.

The following lemma can be proved by straightforward calculation. In particular, (2) is proved in [Nak01, § 10.1].

Lemma 5.11. (1) Under Assumption 5.7(1), we have

$$\begin{aligned} \mathcal{F}_k(v_1 + e_i, w) - q_1 q_2 \mathcal{F}_k(v_1 + e_i, w) - \mathcal{F}_k(v_1, w) + q_1 q_2 \mathcal{F}_k(v_1, w) = \\ \begin{cases} (q_1 q_2 - (q_1 q_2)^{-1}) \mathcal{L}_k, & \text{if } k = i, \\ (q_1^{-1} - q_2) a_{ik} \mathcal{L}_i + (q_2^{-1} - q_1) a_{ki} \mathcal{L}_i, & \text{if } k \neq i \end{cases} \end{aligned}$$

(2) Under Assumption 5.7(2), we have

$$\mathcal{F}_k(v_1 + e_i, w) - q^2 \mathcal{F}_k(v_1 + e_i, w) - \mathcal{F}_k(v_1, w) + q^2 \mathcal{F}_k(v_1, w) = (q^{c_{ki}} - q^{-c_{ki}}) \mathcal{L}_i.$$

We define a \mathcal{P}^0 -action on $\mathcal{M}(w)$ by

$$\Phi_k(z) \cdot m = \frac{\lambda_{-1/z}(\mathcal{F}_k(v, w))}{\lambda_{-1/z}(q_1 q_2 \mathcal{F}_k(v, w))} m,$$

for any $m \in \mathcal{M}(v, w)$.

Proposition 5.12. For any $w \in \mathcal{N}^I$, the action of \mathcal{P} and \mathcal{P}^0 on $\mathcal{M}(w)$ can be extended to the action of $\mathcal{P}^e = \mathcal{P}^0 \ltimes \mathcal{P}$ on $\mathcal{M}(w)$.

Proof. By the same argument as in [Nak01, §10.4], it suffices to show

$$\Phi_k(z) g \Phi_k(z)^{-1}(m) = g \widehat{\Phi}_k(z, v)(m),$$

for any $m \in \mathcal{M}(v_1, w)$, and the simple roots $v = e_i$, $i \in I$. In this case, $g = g(z^{(i)}) \in A_T(\text{pt})[[z^{(i)}]]$. By Theorem 5.6, the action of $g(z^{(i)})$ on $\mathcal{M}(w)$ is given by convolution with $g(c_1(\mathcal{L}_i))$, where \mathcal{L}_i is the tautological line bundle on Hecke correspondence $C_i^+(v_1 + e_i, w)$. The action of $\Phi_k(z) z^{(i)} \Phi_k(z)^{-1}(m)$ is then given by

$$\begin{aligned} & \left(\Delta_* \frac{\lambda_{-1/z}(\mathcal{F}_k(v_1 + e_i, w))}{\lambda_{-1/z}(q_1 q_2 \mathcal{F}_k(v_1 + e_i, w))} \right) \star c_1(\mathcal{L}_i) \left(\Delta_* \frac{\lambda_{-1/z}(\mathcal{F}_k(v_1, w))}{\lambda_{-1/z}(q_1 q_2 \mathcal{F}_k(v_1, w))} \right)^{-1} \\ &= \star c_1(\mathcal{L}_i) \cdot \left(\Delta_* \frac{\lambda_{-1/z}(\mathcal{F}_k(v_1 + e_i, w))}{\lambda_{-1/z}(q_1 q_2 \mathcal{F}_k(v_1 + e_i, w))} \frac{\lambda_{-1/z}(q_1 q_2 \mathcal{F}_k(v_1, w))}{\lambda_{-1/z}(\mathcal{F}_k(v_1, w))} \right) \end{aligned}$$

where the equality is obtained by projection formula.

Under Assumption 5.7(1), by Lemma 5.11(1), the action of $\Phi_k(z) z^{(i)} \Phi_k(z)^{-1}(m)$ coincides with the action of $z^{(i)} \widehat{\Phi}_k(z, e_i)$, where

$$\widehat{\Phi}_k(z, v) = \prod_{i \in I \setminus \{k\}} \prod_{j=1}^{v^i} \frac{(z -_F \lambda_j^{(i)} -_F t_1)^{a_{ik}} (z -_F \lambda_j^{(i)} -_F t_2)^{a_{ki}}}{(z -_F \lambda_j^{(i)} +_F t_2)^{a_{ik}} (z -_F \lambda_j^{(i)} +_F t_1)^{a_{ki}}} \cdot \prod_{j=1}^{v^k} \frac{(z -_F \lambda_j^{(k)} +_F t_1 +_F t_2)}{(z -_F \lambda_j^{(k)} -_F t_1 -_F t_2)}.$$

Similarly, the claim holds in the case under Assumption 5.7(2), by Lemma 5.11(1). \square

5.6. Twisting the preprojective CoHA. In order to relate the spherical subalgebra of the preprojective CoHA to quantum groups, we need to modify the multiplication of the preprojective CoHA by a sign.

As in [Nak01], we define the adjacency matrices AD and \overline{AD} as

$$(AD)_{kl} := \#\{h \in H \mid \text{in}(h) = k, \text{out}(h) = l\},$$

$$(\overline{AD})_{kl} := \#\{h \in H^{\text{op}} \mid \text{in}(h) = k, \text{out}(h) = l\}.$$

Thus, $(AD)^t = \overline{AD}$. We define the matrices C, \overline{C} as

$$(11) \quad C := I - AD, \overline{C} := I - \overline{AD}.$$

Let $m_{v_1, v_2} : \mathcal{P}_{v_1} \otimes \mathcal{P}_{v_2} \rightarrow \mathcal{P}_{v_1 + v_2}$ be the multiplication defined in § 4.

Definition 5.13. The twisted preprojective CoHA, denoted by $\tilde{\mathcal{P}}$, is $\tilde{\mathcal{P}} := \bigoplus_v \mathcal{P}_v$ as \mathbb{N}^I -graded $R[[t_1, t_2]]$ -module, endowed with the multiplication \tilde{m}_{v_1, v_2}

$$\tilde{m}_{v_1, v_2} := (-1)^{(v_2, \overline{C}v_1) + 1} m_{v_1, v_2},$$

where (\cdot, \cdot) is the standard inner product on k^I .

Lemma 5.14. *The multiplication \tilde{m}_{v_1, v_2} is associative.*

Proof. This follows from the associativity of $m^{\mathcal{P}}$:

$$\begin{aligned} \tilde{m}_{v_1 + v_2, v_3}(\tilde{m}_{v_1, v_2}(x_1, x_2), x_3) &= (-1)^{(v_3, \overline{C}(v_1 + v_2)) + 1} (-1)^{(v_2, \overline{C}v_1) + 1} m_{v_1 + v_2, v_3}^{\mathcal{P}}(m_{v_1, v_2}^{\mathcal{P}}(x_1, x_2), x_3) \\ &= (-1)^{(v_2 + v_3, \overline{C}v_1) + 1} (-1)^{(v_3, \overline{C}v_2) + 1} m_{v_1, v_2 + v_3}^{\mathcal{P}}(x_1, m_{v_2, v_3}^{\mathcal{P}}(x_2, x_3)) \\ &= \tilde{m}_{v_1, v_2 + v_3}(x_1, \tilde{m}_{v_2, v_3}(x_2, x_3)). \end{aligned}$$

□

Similarly, we define $\widetilde{\mathcal{SH}}$ to be \mathcal{SH} as \mathbb{N}^I -graded R -module, with multiplication given by

$$\tilde{m}_{v_1, v_2} := (-1)^{(v_2, \overline{C}v_1) + 1} m_{v_1, v_2},$$

where m_{v_1, v_2} is the multiplication of \mathcal{SH} .

For any $w \in \mathbb{N}^I$ we define the map, for each $v_1, v_2 \in \mathbb{N}^I$,

$$\tilde{a}_{v_1, v_2} := (-1)^{(v_2, \overline{C}v_1) + 1} a_{v_1, v_2} : \mathcal{M}(v_1, w) \otimes \tilde{\mathcal{P}}_{v_2} \rightarrow \mathcal{M}(v_1 + v_2, w).$$

Lemma 5.15. *Notations are as above.*

- (1) *There is a well-defined algebra homomorphism $\tilde{\mathcal{P}} \rightarrow \widetilde{\mathcal{SH}}$.*
- (2) *The maps \tilde{a}_{v_1, v_2} define an action of $\tilde{\mathcal{P}}$ on $\mathcal{M}(w)$.*

As in the untwisted case, we write $\tilde{a}_v : \mathcal{P}_v \rightarrow \bigoplus_{v_1 \in \mathbb{N}^I} \text{Hom}(\mathcal{M}(v_1, w), \mathcal{M}(v_1 + v, w))$.

Recall that $\mathcal{P}_{e_k} := A_{G_{e_k} \times T}(\mu_{e_k}^{-1}(0)) = A_{\mathbb{G}_m \times T}(\text{pt}) \cong A_T(\text{pt})[[z^{(k)}]]$. By Theorem 5.6, the action of $(z^{(k)})^l \in \mathcal{P}_{e_k} \subseteq \tilde{\mathcal{P}}$ on the Nakajima quiver varieties $\mathcal{M}(w) := \bigoplus_v \mathcal{M}(v, w)$ is by

$$(12) \quad \tilde{a}_{e_k}((z^{(k)})^l) \mapsto \sum_v (-1)^{(e_k, \overline{C}(v))} (c_1(\mathcal{L}_k))^l \star,$$

where \mathcal{L}_k is the tautological line bundle on the Hecke correspondence $C_k^+(v, w)$, and \star is convolution action.

We define the *spherical subalgebra* of $\tilde{\mathcal{P}}$, denoted by $\tilde{\mathcal{P}}^s$, to be the subalgebra generated by \mathcal{P}_{e_k} , for $k \in I$.

Remark 5.16. The image of $\widetilde{\mathcal{P}^{s,\epsilon}}$ in $\widetilde{\mathcal{SH}}^\epsilon$ is a deformation of $U(\mathfrak{b}_Q[[u]]) \subseteq U(\mathfrak{g}_Q[[u]])$ associated to the formal group law of the OCT A . In the case when $A = \text{CH}$, Theorem 7.3 below shows that this deformation is the same as the Borel of the Yangian. When $A = K$, according to Grojnowski's work [Gr94b], this deformation is expected to be the Borel of the quantum loop algebra.

6. YANGIANS AND SHUFFLE ALGEBRAS

From this point on we have several miscellaneous sections. For any quiver Q without edge-loops, and $A = \text{CH}$, in this section, we show there is a map from the Yangian to the (twisted) shuffle algebra. As will be shown in § 7, this map is injective.

Through out this section, we assume the quiver Q has no edge-loops. We assume the T -action has the same weights as in Remark 3.2.

6.1. The Yangian. Let \mathfrak{g}_Q be the symmetric Kac-Moody Lie algebra associated to the quiver Q . The Cartan matrix of \mathfrak{g}_Q is $C + \overline{C} = (c_{kl})_{k,l \in I}$, which is a symmetric matrix. Recall that the Yangian of \mathfrak{g}_Q , denoted by $Y_\hbar(\mathfrak{g}_Q)$, is an associative algebra over $\mathbb{C}[[\hbar]]$, generated by the variables

$$x_{k,r}^\pm, h_{k,r}, (k \in I, r \in \mathbb{N}),$$

subject to certain relations. Let $Y_\hbar^+(\mathfrak{g}_Q)$ be the subalgebra of $Y_\hbar(\mathfrak{g}_Q)$ generated by the elements $x_{k,r}^+$, for $k \in I, r \in \mathbb{N}$. Define the generating series $x_k^+(u) \in Y_\hbar(\mathfrak{g}_Q)[[u^{-1}]]$ by $x_k^+(u) = \hbar \sum_{r \geq 0} x_{k,r}^+ u^{-r-1}$. The following is a complete set of relations defining $Y_\hbar^+(\mathfrak{g}_Q)$:

$$\begin{aligned} \text{(Y1)} \quad & (u - v - \frac{\hbar c_{kl}}{2}) x_k^+(u) x_l^+(v) = (u - v + \frac{\hbar c_{kl}}{2}) x_l^+(v) x_k^+(u) \\ & + \hbar \left([x_{k,0}^+, x_l^+(v)] - [x_k^+(u), x_{l,0}^+] \right), \text{ for any } k, l \in I. \\ \text{(Y2)} \quad & \sum_{\sigma \in \mathfrak{S}_{1-c_{kl}}} [x_k^+(u_{\sigma(1)}), [x_k^+(u_{\sigma(2)}), [\cdots, [x_k^+(u_{\sigma(1-c_{kl})}), x_l^+(v)] \cdots]]] = 0, \text{ for } k \neq l \in I. \end{aligned}$$

6.2. The map from Yangian to the shuffle algebra. In this subsection, we assume (R, F) is the additive formal group law. We prove the following.

Theorem 6.1. *Let (R, F) be the additive formal group law. Let Q be any quiver without edge loops. The assignment*

$$Y_\hbar^+(\mathfrak{g}) \ni x_{k,r}^+ \mapsto (\lambda^{(k)})^r \in \mathcal{SH}_{e_k} = R[t_1, t_2][\lambda^{(k)}]$$

extends to a well-defined algebra homomorphism $Y_\hbar^+(\mathfrak{g}) \rightarrow \widetilde{\mathcal{SH}}|_{t_1=t_2=\hbar}$.

In order to prove Theorem 6.1, we need to verify the relations (Y1) and (Y2) in the algebra $\widetilde{\mathcal{SH}}$. It will take the rest of § 6.2.

For simplicity, we write the multiplication in $\widetilde{\mathcal{SH}}$ as $*$.

6.2.1. The quadratic relation (Y1). We now check the relation (Y1) in the shuffle algebra. We have

$$x_k^+(u) \mapsto \hbar \sum_{r \geq 0} (\lambda^{(k)})^r u^{-r-1} = \frac{\hbar}{u - \lambda^{(k)}},$$

where the equality is understood in the usual sense. To check the above quadratic relation (Y1), it suffices to show

$$(13) \quad \begin{aligned} & (u - v - \frac{\hbar c_{kl}}{2}) \frac{\hbar}{u - \lambda^{(k)}} * \frac{\hbar}{v - \lambda^{(l)}} - (u - v + \frac{\hbar c_{kl}}{2}) \frac{\hbar}{v - \lambda^{(l)}} * \frac{\hbar}{u - \lambda^{(k)}} \\ &= \hbar \left(1^{(k)} * \frac{\hbar}{v - \lambda^{(l)}} - \frac{\hbar}{v - \lambda^{(l)}} * 1^{(k)} - \frac{\hbar}{u - \lambda^{(k)}} * 1^{(l)} + 1^{(l)} * \frac{\hbar}{u - \lambda^{(k)}} \right). \end{aligned}$$

We first consider the case when $k \neq l$. For simplicity, we write $a = -c_{kl}$. Let S be the set $\{a, a-2, a-4, \dots, -a+4, -a+2\}$. By the shuffle formula (5), the multiplication $\widetilde{\mathcal{SH}}_{e_k} \otimes \widetilde{\mathcal{SH}}_{e_l} \rightarrow \widetilde{\mathcal{SH}}_{e_k+e_l}$ is given by

$$(\lambda^{(k)})^p * (\lambda^{(l)})^q = -(\lambda^{(k)})^p (\lambda^{(l)})^q \prod_{m \in S} (\lambda^{(l)} - \lambda^{(k)} + m \frac{\hbar}{2}).$$

Similarly, the multiplication $\widetilde{\mathcal{SH}}_{e_l} \otimes \widetilde{\mathcal{SH}}_{e_k} \rightarrow \widetilde{\mathcal{SH}}_{e_k+e_l}$ is given by

$$(\lambda^{(l)})^p * (\lambda^{(k)})^q = (-1)^{a+1} (\lambda^{(l)})^p (\lambda^{(k)})^q \prod_{m \in S} (\lambda^{(k)} - \lambda^{(l)} + m \frac{\hbar}{2}) = -(\lambda^{(l)})^p (\lambda^{(k)})^q \prod_{m \in S} (\lambda^{(l)} - \lambda^{(k)} - m \frac{\hbar}{2}).$$

Plugging into equation (13), (Y1) becomes the following identity

$$(14) \quad \begin{aligned} & (u - v - \frac{\hbar c_{kl}}{2}) \frac{\hbar}{u - \lambda^{(k)}} \frac{\hbar}{v - \lambda^{(l)}} \prod_{m \in S} (\lambda^{(l)} - \lambda^{(k)} + m \frac{\hbar}{2}) \\ & - (u - v + \frac{\hbar c_{kl}}{2}) \frac{\hbar}{v - \lambda^{(l)}} \frac{\hbar}{u - \lambda^{(k)}} \prod_{m \in S} (\lambda^{(l)} - \lambda^{(k)} - m \frac{\hbar}{2}) \\ &= \hbar \left(\frac{\hbar}{v - \lambda^{(l)}} \prod_{m \in S} (\lambda^{(l)} - \lambda^{(k)} + m \frac{\hbar}{2}) - \frac{\hbar}{v - \lambda^{(l)}} \prod_{m \in S} (\lambda^{(l)} - \lambda^{(k)} - m \frac{\hbar}{2}) \right. \\ & \quad \left. - \frac{\hbar}{u - \lambda^{(k)}} \prod_{m \in S} (\lambda^{(l)} - \lambda^{(k)} + m \frac{\hbar}{2}) + \frac{\hbar}{u - \lambda^{(k)}} \prod_{m \in S} (\lambda^{(l)} - \lambda^{(k)} - m \frac{\hbar}{2}) \right). \end{aligned}$$

Canceling the common factor

$$\prod_{m \in S'} (\lambda^{(l)} - \lambda^{(k)} + m \frac{\hbar}{2}), \text{ where } S' = \{a-2, a-4, \dots, -a+2\},$$

The equality (14) becomes

$$(u - v + \frac{\hbar a}{2}) \frac{\lambda^{(l)} - \lambda^{(k)} + a \frac{\hbar}{2}}{(u - \lambda^{(k)})(v - \lambda^{(l)})} - (u - v - \frac{\hbar a}{2}) \frac{\lambda^{(l)} - \lambda^{(k)} - a \frac{\hbar}{2}}{(v - \lambda^{(l)})(u - \lambda^{(k)})} = a \hbar \left(\frac{1}{v - \lambda^{(l)}} - \frac{1}{u - \lambda^{(k)}} \right).$$

Both sides of the above identity are equal to $a \hbar \frac{u-v+\lambda^{(l)}-\lambda^{(k)}}{(u-\lambda^{(k)})(v-\lambda^{(l)})}$. This shows the relation (Y1) for the case when $k \neq l$.

We now check the relation (Y1) when $k = l$. A similar calculation using the shuffle formula (5) shows that equation (13) becomes the following identity in $\widetilde{\mathcal{SH}}_{2e_k} = R[t_1, t_2][\lambda_1, \lambda_2]$

$$\begin{aligned} & (u - v - \hbar) \sum_{\sigma \in \mathfrak{S}_2} \sigma \left(\frac{\hbar}{u - \lambda_1} \frac{\hbar}{v - \lambda_2} \frac{\lambda_1 - \lambda_2 + \hbar}{\lambda_2 - \lambda_1} \right) - (u - v + \hbar) \sum_{\sigma \in \mathfrak{S}_2} \sigma \left(\frac{\hbar}{v - \lambda_1} \frac{\hbar}{u - \lambda_2} \frac{\lambda_1 - \lambda_2 + \hbar}{\lambda_2 - \lambda_1} \right) \\ &= \hbar \sum_{\sigma \in \mathfrak{S}_2} \sigma \left(\left(\frac{\hbar}{v - \lambda_2} - \frac{\hbar}{v - \lambda_1} - \frac{\hbar}{u - \lambda_1} + \frac{\hbar}{u - \lambda_2} \right) \frac{\lambda_1 - \lambda_2 + \hbar}{\lambda_2 - \lambda_1} \right). \end{aligned}$$

It is straightforward to show that both sides of the above identity can be simplified to

$$\frac{2\hbar^3}{\lambda_2 - \lambda_1} \left(\frac{1}{v - \lambda_2} - \frac{1}{v - \lambda_1} - \frac{1}{u - \lambda_1} + \frac{1}{u - \lambda_2} \right).$$

This completes the proof of relation (Y1) for $k = l$.

6.2.2. *The Serre relation (Y2).* An argument similar to [Nak01, § 10.4] shows that to check the relation (Y2), it suffices to check:

$$(Y2') \quad \sum_{p=0}^{1-c_{kl}} (-1)^p \binom{1-c_{kl}}{p} x_{k,0}^{*p} * x_{l,0} * x_{k,0}^{*(1-c_{kl}-p)} = 0,$$

where $x^{*n} = x * x * \cdots * x$, the shuffle product of n -copies of x . We use the shuffle formula (5) to check the Serre relation (Y2').

For any i, j , let $\lambda_{i,j} = \lambda_i - \lambda_j$. By the shuffle formula (5), we have the recurrence relation

$$(x_{k,0})^{*n+1} = (-1)^{n+1} \sum_{\sigma \in \text{Sh}_{(n,1)}} \sigma \left((x_{k,0})^{*n} \prod_{i=1}^n \frac{\lambda_{i,n+1}^{(k)} + \hbar}{\lambda_{n+1,i}^{(k)}} \right).$$

Therefore, inductively, we get a formula of $(x_{k,0})^{*n}$:

$$(15) \quad (x_{k,0})^{*n} = (-1)^{\frac{n(n+1)}{2}-1} \sum_{\sigma \in \mathfrak{S}_n} \sigma \left(\frac{\lambda_{12}^{(k)} + \hbar}{\lambda_{21}^{(k)}} \cdot \frac{\lambda_{13}^{(k)} + \hbar}{\lambda_{31}^{(k)}} \cdots \frac{\lambda_{n-1,n}^{(k)} + \hbar}{\lambda_{n,n-1}^{(k)}} \right).$$

Note that $k \neq l$. By the shuffle formula (5), the multiplication $\widetilde{\mathcal{SH}}_{ne_k} \otimes \widetilde{\mathcal{SH}}_{e_l} \rightarrow \widetilde{\mathcal{SH}}_{ne_k+e_l}$ is given by

$$(16) \quad (x_{k,0})^{*n} * x_{l,0} = -(x_{k,0})^{*n} \prod_{i=1}^n \prod_{m \in S} (\lambda^{(l)} - \lambda_i^{(k)} + m \frac{\hbar}{2}),$$

where $S = \{a, a-2, a-4, \dots, -a+4, -a+2\}$.

For the multiplication $\widetilde{\mathcal{SH}}_{pe_k+e_l} \otimes \widetilde{\mathcal{SH}}_{qe_k} \rightarrow \widetilde{\mathcal{SH}}_{e_l+(p+q)e_k}$, considered as a map

$$R[t_1, t_2][\lambda_1^{(k)}, \dots, \lambda_p^{(k)}, \lambda_1^{(l)}] \otimes R[t_1, t_2][\lambda_{p+1}^{(k)}, \dots, \lambda_{p+q}^{(k)}] \rightarrow R[t_1, t_2][\lambda_1^{(k)}, \dots, \lambda_{p+q}^{(k)}, \lambda_1^{(l)}],$$

we have

$$(17) \quad \begin{aligned} ((x_{k,0})^{*p} * x_{l,0}) * (x_{k,0})^{*q} &= (-1)^{pq+qa+1} \sum_{\sigma \in \text{Sh}_{(p,q)}} \sigma \left((x_{k,0})^{*p} * x_{l,0} \cdot (x_{k,0})^{*q} \right. \\ &\quad \cdot \prod_{s=1}^p \prod_{t=p+1}^{p+q} \frac{\lambda_s^{(k)} - \lambda_t^{(k)} + \hbar}{\lambda_t^{(k)} - \lambda_s^{(k)}} \cdot \left. \prod_{t=p+1}^{p+q} \prod_{m \in S} (\lambda_t^{(k)} - \lambda^{(l)} + m \frac{\hbar}{2}) \right) \end{aligned}$$

Plugging the formulas of (15) (16) into (17) with $q = a + 1 - p$, we get

$$\begin{aligned} x_{k,0}^{*p} * x_{l,0} * x_{k,0}^{*(a+1-p)} &= (-1)^{\frac{(a+1)(a+2)}{2}} \sum_{\pi \in \text{Sh}_{(p,a+1-p)}} \pi \left(\left(\sum_{\sigma \in \mathfrak{S}_p} \sigma \cdot \prod_{1 \leq i < j \leq p} \frac{\lambda_{i,j}^{(k)} + \hbar}{\lambda_{j,i}^{(k)}} \right) \right. \\ &\quad \cdot \left(\sum_{\sigma \in \mathfrak{S}_{a+1-p}} \sigma \cdot \prod_{\{p+1 \leq i < j \leq a+1\}} \frac{\lambda_{i,j}^{(k)} + \hbar}{\lambda_{j,i}^{(k)}} \right) \left(\prod_{s=1}^p \prod_{t=p+1}^{a+1} \frac{\lambda_{s,t}^{(k)} + \hbar}{\lambda_{t,s}^{(k)}} \right) \\ &\quad \cdot \left. \prod_{m \in S} \left(\prod_{i=1}^p (\lambda_i^{(k)} - \lambda^{(l)} - m \frac{\hbar}{2}) \prod_{t=p+1}^{a+1} (\lambda_t^{(k)} - \lambda^{(l)} + m \frac{\hbar}{2}) \right) \right). \end{aligned}$$

Re-arranging the above summation, we have:

$$\begin{aligned} \sum_{p=0}^{a+1} (-1)^p \binom{a+1}{p} x_{k,0}^{*p} * x_{l,0} * x_{k,0}^{*(a+1-p)} &= \sum_{p=0}^{a+1} (-1)^p \binom{a+1}{p} \sum_{\sigma \in \mathfrak{S}_{a+1}} \left(\prod_{1 \leq i < j \leq a+1} \frac{\lambda_{\sigma(i),\sigma(j)}^{(k)} + \hbar}{\lambda_{\sigma(j),\sigma(i)}^{(k)}} \right. \\ &\quad \cdot \left. \prod_{m \in S} \left(\prod_{i=1}^p (\lambda_{\sigma(i)}^{(k)} - \lambda^{(l)} - m \frac{\hbar}{2}) \prod_{t=p+1}^{a+1} (\lambda_{\sigma(t)}^{(k)} - \lambda^{(l)} + m \frac{\hbar}{2}) \right) \right). \end{aligned}$$

Note that the factor

$$\prod_{m \in S'} \left(\prod_{i=1}^p (\lambda_{\sigma(i)}^{(k)} - \lambda^{(l)} - m \frac{\hbar}{2}) \prod_{t=p+1}^{a+1} (\lambda_{\sigma(t)}^{(k)} - \lambda^{(l)} + m \frac{\hbar}{2}) \right) = \prod_{m \in S'} \prod_{i=1}^{a+1} (\lambda_i^{(k)} - \lambda^{(l)} - m \frac{\hbar}{2}),$$

is independent of $\sigma \in \mathfrak{S}_{a+1}$, hence a common factor. Here again $S' = \{a-2, a-4, \dots, -a+2\}$. Let $\lambda_i^{(k)} = \lambda_i^{(k)} - \lambda^{(l)}$. After canceling the above common factor, to show the Serre relation (Y2'), it suffices to show

$$(18) \quad \sum_{p=0}^{a+1} (-1)^p \binom{a+1}{p} \sum_{\sigma \in \mathfrak{S}_{a+1}} \sigma \left(\prod_{s=1}^p (\lambda_s^{(k)} - \frac{a\hbar}{2}) \prod_{t=p+1}^{a+1} (\lambda_t^{(k)} + \frac{a\hbar}{2}) \prod_{1 \leq i < j \leq a+1} \frac{\lambda_{i,j}^{(k)} + \hbar}{\lambda_{j,i}^{(k)}} \right) = 0.$$

The identity (18) is proved in Appendix, Corollary A.2. As a consequence, the map $Y_{\hbar}^+(\mathfrak{g}) \rightarrow \widetilde{\mathcal{SH}}|_{t_1=t_2=\hbar}$ respects the Serre relation (Y2).

In Theorem A.1, we have a more general identity of symmetric functions, which deforms the identity (18). It might be interesting on its own rights. Therefore, we prove it in Appendix A and deduce (18) from it.

7. COMPARISON WITH YANGIAN ACTIONS ON QUIVER VARIETIES

In this section, we show that the (twisted) spherical subalgebra $\mathcal{P}^{\mathfrak{s}, \epsilon}$ of preprojective CoHA defined in § 4 specializes to the Borel of the Yangian when A is the intersection theory. Again we assume Q is a quiver without edge-loops, and the T -action has the same weights as in Remark 3.2.

7.1. Comparison of the preprojective CoHA with Varagnolo Yangian action. Now we take the oriented Borel-Moore homology theory to be the intersection theory CH. Recall that Varagnolo in [Va00] constructed representations of the Yangians using quiver variety. It is proved

that, for each $w \in \mathbb{N}^I$, there is an algebra homomorphism $a^Y : Y_h(\mathfrak{g}) \rightarrow \text{End}(\text{CH}_{G_w \times \mathbb{G}_m}(\mathfrak{M}(w)))$. The action of generator $x_{k,r}^+$ is given by

$$x_{k,r}^+ \mapsto \sum_{v_2} (-1)^{(e_k, \overline{C}v_2)} \Delta_*^+(c_1(\mathcal{L}_k))^r \in \text{CH}_{G_w \times T}(Z(w)) \rightarrow \text{End}(\mathcal{M}(w)),$$

where

$$\Delta^+ : C_k^+(v_2, w) \hookrightarrow Z(v_2 - e_k, v_2, w)$$

is the natural embedding of the irreducible component.¹

Observe that according to the projection formula and (12), we have $a^Y(x_{k,r}^+) = \tilde{a}((z^{(k)})^l) \in \text{End}(\mathcal{M}(w))$.

Assume the quiver Q is *ADE* type. For $w \in \mathbb{N}^I$, we call the action map $a_w^Y : Y_h(\mathfrak{g}) \rightarrow \text{End}(\mathcal{M}(w))$ a_w^Y to emphasize the dependence on w .

Lemma 7.1.² *With notations as above, we have $\bigcap_w \ker(a_w^Y) = 0$.*

Proof. In [Nak12], Nakajima proved that for any $w_1, w_2 \in \mathbb{N}^I$, the kernel of the map $Y_h(\mathfrak{g}) \rightarrow \text{End}(\mathcal{M}(w_1) \otimes \mathcal{M}(w_2))$ is contained in the kernel of $a_{w'}^Y$ for some $w' \in \mathbb{N}^I$ (see also the proof of Lemma 7.6). Therefore, the ideal $\bigcap_w \ker(a_w^Y)$ is a $\mathbb{C}[\hbar]$ -flat Hopf ideal in $Y_h(\mathfrak{g})$. By [GTL10, Proposition A.8], if Q is of finite Dynkin type, there is no non-trivial such ideal in $Y_h(\mathfrak{g})$. Therefore, $\bigcap_w \ker(a_w^Y) = 0$. \square

The following is a direct consequence of Lemma 7.1.

Corollary 7.2. *Assume Q is a quiver of ADE type. The assignment $(z^{(k)})^l \mapsto x_{k,l}^+$, $t_1, t_2 \mapsto \hbar/2$ extends to a well-defined surjective algebra homomorphism $\Upsilon : {}^{\text{CH}}\tilde{\mathcal{P}}^s \rightarrow Y_h^+(\mathfrak{g})$. Moreover, the following diagram commutes*

$$\begin{array}{ccc} {}^{\text{CH}}\tilde{\mathcal{P}}^s & \xrightarrow{\tilde{a}} & \text{End}(\text{CH}_{G_w \times \mathbb{G}_m}(\mathfrak{M}(w))) \\ \Upsilon \downarrow & & \uparrow a^Y \\ Y_h^+(\mathfrak{g}) & \hookrightarrow & Y_h(\mathfrak{g}). \end{array}$$

For an *ADE* type quiver Q , summarizing Lemma 5.15, Theorem 6.1, and Corollary 7.2, we have a commutative diagram of algebras

$$\begin{array}{ccc} {}^{\text{CH}}\tilde{\mathcal{P}}^s & \longrightarrow & \widetilde{\mathcal{SH}}|_{t_1=t_2=\hbar/2} \\ \Upsilon \downarrow & \nearrow & \\ Y_h^+(\mathfrak{g}) & & \end{array}$$

For any $x \in {}^{\text{CH}}\tilde{\mathcal{P}}^s|_{t_1=t_2=\hbar/2}$ such that $\Upsilon(x) = 0$, then x lies in the kernel of the map ${}^{\text{CH}}\tilde{\mathcal{P}}^s|_{t_1=t_2=\hbar/2} \rightarrow \widetilde{\mathcal{SH}}^s|_{t_1=t_2=\hbar/2}$. We know this map is an isomorphism after localization, i.e., inverting \hbar . Therefore, x is a torsion element in ${}^{\text{CH}}\tilde{\mathcal{P}}^s|_{t_1=t_2=\hbar/2}$.

Define ${}^{\text{CH}}\tilde{\mathcal{P}}^s_{\text{tors}}$ to be the quotient of ${}^{\text{CH}}\tilde{\mathcal{P}}^s|_{t_1=t_2=\hbar/2}$ by the \hbar -torsion part.

¹In [Va00] the Borel-Moore homology was used instead of the intersection theory. However, note that in the verification of Yangian action, one only uses the fact that the formal group law of this cohomology theory is the additive group law.

²We thank Sachin Gautam for explaining to us the proof of [GTL10, Proposition A.8].

Theorem 7.3. *Assume Q is a quiver of ADE type. We have the following isomorphism*

$$\Upsilon^{-1} : Y_h^+(\mathfrak{g}) \cong {}^{\text{CH}}\widetilde{\mathcal{P}}^s,$$

such that the diagram

$$\begin{array}{ccc} Y_h^+(\mathfrak{g}) & \xhookrightarrow{\quad} & Y_h(\mathfrak{g}) \\ \Upsilon^{-1} \downarrow & & a^Y \downarrow \\ {}^{\text{CH}}\widetilde{\mathcal{P}}^s & \xrightarrow{\tilde{a}} & \text{End}(\text{CH}_{G_w \times \mathbb{G}_m}(\mathfrak{M}(w))) \end{array}$$

commutes. Here the action a^Y is given by Varagnolo in [Va00].

Remark 7.4. As will be shown in § 7.3, Theorem 7.3 is true for any quiver Q without edge loops.

7.2. Reminder on the Maulik-Okounkov Yangian. From now on till the end of § 7, we assume the base field of Q -representations to be $k = \mathbb{C}$. We consider the Chow group with \mathbb{C} coefficients. In [MO12], Maulik-Okounkov constructed a bigger Yangian Y_{MO} , acting on the homology of Nakajima quiver variety $\mathcal{M}(w)$. We compare the preprojective CoHA action with the Maulik-Okounkov Yangian action.

We briefly recall the construction in [MO12]. Let $A = \{(z_1, \dots, z_n) \mid z_i \in \mathbb{G}_m\}$ be a torus. We have a map $A \rightarrow G_w$ given by

$$(z_1, \dots, z_n) \mapsto \text{diag}(z_1 \text{id}_{w^{(1)}}, \dots, z_n \text{id}_{w^{(n)}}),$$

where $w = \sum_{i=1}^n z_i w^{(i)}$. We have the decomposition of A -fixed points of $\mathfrak{M}(w)$:

$$\mathfrak{M}(w)^A = \mathfrak{M}(w^{(1)}) \times \dots \times \mathfrak{M}(w^{(n)}).$$

Recall that (see [MO12, Page 111]) the Maulik-Okounkov Yangian Y_{MO} is the subalgebra of

$$\prod_n \prod_{i_1, \dots, i_n} \text{End}_{\mathbb{C}[u_1, \dots, u_n]}(F_{i_1}(u_1) \otimes \dots \otimes F_{i_n}(u_n)),$$

where $F_i(u_i) := \text{CH}_{G_A}(\mathfrak{M}(w^{(i)})) \otimes \mathbb{C}[u_i]$, for $G_A = G_{w^{(1)}} \times \dots \times G_{w^{(n)}}$. Choose an additional $F_0 \in$ the set $\{F_i\}$ called an *auxiliary* space and define

$$R_{F_0(u), W} := R_{F_0(u), F_n}(u - u_n) \cdots R_{F_0(u), F_1}(u - u_1),$$

for $W = F_1(u_1) \otimes \dots \otimes F_n(u_n)$. See [MO12, Page 79] for the construction of R -matrix. Let $F_0^\vee := \text{Hom}(F_0, \mathbb{C})$ be the graded dual module. Consider the polynomial in u

$$m(u) \in F_0 \otimes F_0^\vee \otimes \mathbb{C}[u]$$

with values in operators in F_0 of finite rank. Because $m(u)$ has finite rank and \hbar divides $R - 1$, the following operator

$$E(m(u)) = -\frac{1}{\hbar} \text{Res}_{u=\infty} \text{tr}_{F_0} m(u) R_{F_0(u), W} \in \text{End}(W)$$

is well defined for all W . It depends polynomially on u_1, \dots, u_n , since it comes from an expansion of rational functions of $u - u_i$ as $u \rightarrow \infty$. By definition, the Maulik-Okounkov Yangian Y_{MO} is the subalgebra generated by 1 and $E(m(u))$ for all F_0 and all $m(u)$.

In particular, for any $w \in \mathbb{N}^I$, we have a map $a_{\text{MO}} : Y_{\text{MO}} \rightarrow \text{End}(\mathcal{M}(w))$ obtained by composing with the projection to $n = 1$ summand. This gives the action of Y_{MO} on $\mathcal{M}(w)$.

There is a Lie algebra $\mathfrak{g}_{\text{MO}} \subset Y_{\text{MO}}$, which is spanned by the operators $E(m_0)$, where m_0 is constant polynomial in u .

In [MO12, Page 123], the algebra of *classical multiplications*

$$\text{Classical} \subset Y_{\text{MO}} \subset \prod_n \prod_{i_1, \dots, i_n} \text{End}_{\mathbb{C}[u_1, \dots, u_n]}(F_{i_1}(u_1) \otimes \dots \otimes F_{i_n}(u_n))$$

is defined to be the subalgebra generated by the characteristic classes of $\{\mathcal{V}_i, \mathcal{W}_i\}$.

Theorem 7.5. [MO12, Theorem 5.5.1] *The Yangian Y_{MO} is generated by the Lie algebra \mathfrak{g}_{MO} and the operators of classical multiplication. Moreover, there is a filtration such that*

$$\text{gr}(Y_{\text{MO}}) \cong U(\mathfrak{g}_{\text{MO}}[u]).$$

Lemma 7.6. *The map $Y_{\text{MO}} \rightarrow \prod_w \text{End}(\mathcal{M}(w))$ is injective.*

Proof. For any torus A , we have a map

$$\text{End}(\mathcal{M}(w)) \rightarrow \text{End}(\text{CH}_{G_A}(\mathfrak{M}(w)^A))$$

constructed in [MO12, Page 126], which is compatible with the comultiplication of the Yangian $\Delta : Y_{\text{MO}} \rightarrow Y_{\text{MO}} \hat{\otimes} Y_{\text{MO}}$. In other words, we have the following commutative diagram

$$\begin{array}{ccc} Y_{\text{MO}} & \hookrightarrow & \\ \downarrow & \searrow & \\ \prod_w \text{End}(\mathcal{M}(w)) & \rightarrow & \prod_w \prod_A \text{End}(\text{CH}_{G_A}(\mathfrak{M}(w)^A)) \end{array}$$

The injectivity of $Y_{\text{MO}} \hookrightarrow \prod_w \prod_A \text{End}(\text{CH}_{G_A}(\mathfrak{M}(w)^A))$ implies the injectivity of the map $Y_{\text{MO}} \rightarrow \prod_w \text{End}(\mathcal{M}(w))$. \square

7.3. Comparison with Maulik-Okounkov Yangian action. Define the filtration on ${}^{\text{CH}}\tilde{\mathcal{P}}^s$ by $\deg((z^{(k)})^l) := l$, and

$${}^{\text{CH}}\tilde{\mathcal{P}}^{s,i} * {}^{\text{CH}}\tilde{\mathcal{P}}^{s,j} \subset {}^{\text{CH}}\tilde{\mathcal{P}}^{s,i+j}.$$

For any quiver Q , let \mathfrak{g}_Q be the Kac-Moody Lie algebra associated to Q , and $\mathfrak{n}_Q \subset \mathfrak{g}_Q$ be the nilpotent subalgebra. Let $Y_h^+(\mathfrak{g}_Q)$ be the positive half of the Yangian defined in § 6.1. The generator of $Y_h^+(\mathfrak{g}_Q)$ are $x_{k,r}^+$, for $k \in I, r \in \mathbb{N}$, which satisfy relations (Y1), (Y2). Let $Y_h^{\geq 0}(\mathfrak{g}_Q)$ be the Borel part of the $Y_h(\mathfrak{g}_Q)$, generated by $Y_h^+(\mathfrak{g}_Q)$ and $h_{i,k}$ for $i \in I$ and $k \in \mathbb{N}$, subject to

$$\begin{aligned} [h_{i,0}, x_{j,s}^+] &= c_{ij} x_{j,s}^+, \\ [h_{i,r+1}, x_{j,s}^+] - [h_{i,r}, x_{j,s+1}^+] &= \frac{c_{ij} \hbar}{2} (h_{i,r} x_{j,s}^+ + x_{j,s}^+ h_{i,r}). \end{aligned}$$

We now state the main theorem of this section.

Theorem 7.7. *Assume quiver Q has no edge loops.*

- (1) *There is an algebra homomorphism ${}^{\text{CH}}\tilde{\mathcal{P}}^{s,\epsilon} \rightarrow Y_{\text{MO}}$, such that the following diagram commutes.*

$$\begin{array}{ccc} {}^{\text{CH}}\tilde{\mathcal{P}}^{s,\epsilon} & \xrightarrow{\Gamma} & Y_{\text{MO}} \\ \tilde{a} \searrow & & \swarrow a_{\text{MO}} \\ & \prod_w \text{End}(\mathcal{M}(w)) & \end{array}$$

- (2) *The map Γ preserves the filtration.*
 (3) *The map Γ is injective, whose image is the Borel subalgebra of the Kac-Moody Yangian $Y_h^{\geq 0}(\mathfrak{g}_Q) \subset Y_{\text{MO}}^{\geq 0} \subset Y_{\text{MO}}$.*

The rest of this section is devoted to the proof of Theorem 7.7.

7.3.1. *Well-definition of Γ .* Recall that $\{(z^{(k)})^l\}_{k \in I, l \in \mathbb{N}}$ are generators of \mathcal{P}^s . By the injectivity of a_{MO} , to show Theorem 7.7(1), it suffices to check

$$\tilde{a}(\mathcal{P}^0) \subset \text{Im}(a_{\text{MO}}), \text{ and } \tilde{a}((z^{(k)})^l) \in \text{Im}(a_{\text{MO}}) \text{ for any } k \in I, l \in \mathbb{N}.$$

Let $X_{k,l} := \tilde{a}((z^{(k)})^l)$ be the image of $(z^{(k)})^l$ in $\text{End}(\mathcal{M}(w))$.

Recall that for any $k \in I$, $\mathcal{F}_k(v, w)$ is an element in $K^{G_w \times T}(\mathfrak{M}(v, w))$ (see § 5.5). Let $\Delta : \mathfrak{M}(v, w) \rightarrow \mathfrak{M}(v, w) \times \mathfrak{M}(v, w)$ is the diagonal embedding and set $\hbar = c_1(q^2)$. Following [Va00, (4.2)], we define $H_{k,r} \in \text{End}(\mathcal{M}(w))$ to be the coefficient of $\hbar z^{-r-1}$ in

$$(19) \quad \left(-1 + \sum_v \Delta_* \frac{\lambda_{-1/z}(\mathcal{F}_k(v, w))}{\lambda_{-1/z}(q^2 \mathcal{F}_k(v, w))} \right)^-,$$

where $-$ stands for the expansion at $z = \infty$ and $\lambda_z(\mathcal{F})$ is the equivariant Chern polynomial. In particular, we have $\tilde{a}(\Phi_k(z)) = 1 + \sum_{r \geq 0} H_{k,r} z^{-r-1}$ for any $k \in I$.

Theorem 7.8. [Va00] *The following relations hold in $\text{End}(\mathcal{M}(w))$*

$$[H_{i,0}, X_{j,s}] = c_{ij} X_{j,s}, \quad [H_{i,r+1}, X_{j,s}] - [H_{i,r}, X_{j,s+1}] = \frac{c_{ij} \hbar}{2} (H_{i,r} X_{j,s} + X_{j,s} H_{i,r}).$$

As a corollary, we have

$$(20) \quad X_{i,s+1} = \frac{1}{2} [H_{i,1}, X_{i,s}] - \frac{\hbar}{2} (H_{i,0} X_{i,s} + X_{i,s} H_{i,0}).$$

Therefore, to show $X_{i,s} \in \text{Im}(a_{\text{MO}})$, we only need to show

$$X_{i,0}, H_{i,r} \in \text{Im}(a_{\text{MO}}), \text{ for any } i \in I \text{ and } r \in \mathbb{N}.$$

By definition, $X_{i,0}$ is given by convolution with the Hecke correspondence $[C_i^+]$. From [M13, Page 33], the operator that convolves with $[C_i^+]$ lies in $\text{Im}(a_{\text{MO}})$. Therefore, $X_{i,0} \in \text{Im}(a_{\text{MO}})$.

By (19), $H_{i,r}$ lies in the subalgebra $\text{Classical} \subset Y_{\text{MO}}$. This completes the proof of Theorem 7.7(1).

7.3.2. *Preservation of the filtration.* Recall that the Maulik-Okounkov Yangian Y_{MO} has a filtration such that

$$\deg(E(m(u))) \leq \deg_u(m(u)),$$

$\deg_u 1 = 0$, and $\deg(\mathfrak{g}_{\text{MO}}) = 0$. In particular, $\Gamma(z^{(k)})^0$ has degree zero for any $k \in I$.

Lemma 7.9. *In the Yangian Y_{MO} , we have $\deg(H_{i,r}) \leq r$.*

Proof. By [MO12, Page 125], all characteristic classes of \mathcal{V}_i and \mathcal{W}_i of degree $< r$ are in $Y_{\text{MO}, < r}$. By the definition of $H_{i,r}$ via expansion of (19), taking into account that $\deg(\hbar) = 1$, we conclude that $\deg(H_{i,r}) \leq r$. \square

Using (20), we obtain $\deg(X_{i,r}) \leq r$. As a consequence, the map Γ preserves the filtration.

7.3.3. *Injectivity.* By [Nak98, Theorem 10.2], the top Borel-Moore homology of Nakajima quiver variety $H_{\text{top}}(\mathfrak{M}(w))$, as a $U(\mathfrak{g}_Q)$ module, is the irreducible highest weight module $L(w)$ with highest weight w . Let \ker_w be the kernel of the map $U(\mathfrak{g}_Q) \rightarrow \text{End}(L(w))$, then, $\cap_w \ker_w = 0$. This implies \mathfrak{n}_Q embeds in $\prod_w \text{End}(\mathcal{M}(w))$. Therefore, one has an embedding $\mathfrak{n}_Q \hookrightarrow \mathfrak{n}_{\text{MO}}$.

The morphism Γ induces the map

$$\text{gr } \Gamma : \text{gr}(\text{CH}^{\text{s}, \epsilon} \widetilde{\mathcal{P}}) \rightarrow \text{gr } Y_{\text{MO}} \cong U(\mathfrak{g}_{\text{MO}}[u])$$

between the associated graded algebras. Clearly, $\text{Im}(\text{gr } \Gamma) \subset U(\mathfrak{b}_{\text{MO}}[u])$. Using the facts that $X_{i,r} \in U(\mathfrak{n}_Q[u])$ and the embedding $\mathfrak{n}_Q \subset \mathfrak{n}_{\text{MO}}$, we obtain that the image of $\text{gr } \Gamma$ coincides with $U(\mathfrak{b}_Q[u])$.

Therefore, we have the following commutative diagram.

$$\begin{array}{ccc} \text{gr}(\text{CH}\tilde{\mathcal{P}}^{\mathfrak{s},\epsilon}) & \xrightarrow{\text{gr } \Gamma'} & U(\mathfrak{b}_Q[u]) \\ \downarrow & & \downarrow \\ \text{gr}(Y_{\text{MO}}^{\geq 0}) & \xrightarrow{\cong} & U(\mathfrak{b}_{\text{MO}}[u]) \end{array}$$

Proposition 7.10. *We have an isomorphism of algebras.*

$$\text{gr}(\text{CH}\tilde{\mathcal{P}}^{\mathfrak{s},\epsilon}) := \bigoplus_i \tilde{\mathcal{P}}^{\mathfrak{s},\epsilon,i} / \tilde{\mathcal{P}}^{\mathfrak{s},\epsilon,i-1} \cong U(\mathfrak{b}_Q[u]).$$

Proof. The morphism in Theorem 6.1 induces the map $U(\mathfrak{n}_Q[u]) \rightarrow \text{gr } \widetilde{\mathcal{SH}}$. It makes the diagram commutative:

$$\begin{array}{ccc} \text{gr}(\text{CH}\tilde{\mathcal{P}}^{\mathfrak{s},\epsilon}) & \xrightarrow{\text{gr } \Gamma'} & U(\mathfrak{b}_Q[u]) \\ \downarrow & \swarrow & \\ \text{gr } \widetilde{\mathcal{SH}}^\epsilon & & \end{array}$$

The injectivity of $\text{CH}\tilde{\mathcal{P}}^{\mathfrak{s},\epsilon} \rightarrow \widetilde{\mathcal{SH}}^\epsilon$ implies the injectivity of $\text{gr } \Gamma'$. This completes the proof. \square

By a standard argument, the fact $\text{gr } \Gamma'$ is an isomorphism implies that Γ' is an isomorphism. Therefore, Γ is injective, and $\text{Im}(\Gamma) = Y_h^{\geq 0}(\mathfrak{g}_Q)$.

Corollary 7.11. *The algebra homomorphism $Y_h^+(\mathfrak{g}_Q) \rightarrow \widetilde{\mathcal{SH}}|_{t_1=t_2=\hbar/2}$ in Theorem 6.1 is injective.*

APPENDIX A. A SYMMETRIC POLYNOMIAL IDENTITY

Let $\mathbb{S}(n, b, \hbar) \in \mathbb{Q}[b, \hbar](\lambda_1, \dots, \lambda_n)^{\mathfrak{S}_n}$ be

$$\mathbb{S}(n, b, \hbar) := \sum_{\sigma \in \mathfrak{S}_n} \sigma \cdot \left(\sum_{p=0}^n (-1)^p \binom{n}{p} \prod_{i=1}^p (\lambda_i - b\hbar) \prod_{j=p+1}^n (\lambda_j + b\hbar) \prod_{1 \leq i < j \leq n} \frac{\lambda_{ij} + \hbar}{\lambda_{ji}} \right).$$

Here $\lambda_{i,j} = \lambda_i - \lambda_j$ for any i, j .

Theorem A.1. (1) *The element $\mathbb{S}(n, b, \hbar)$ lies in $\mathbb{Q}[b, \hbar]$. In other words, $\mathbb{S}(n, b, \hbar)$ does not depend on the variables $\{\lambda_i, i = 1, \dots, n\}$.*

(2) *We have $\mathbb{S}(1, b, \hbar) = 2\hbar b$, and the recursive formula of $\mathbb{S}(n, b, \hbar)$*

$$\mathbb{S}(n, b, \hbar) = \mathbb{S}(n-1, b, \hbar) \cdot \left((-1)^{n+1} 2\hbar n \left(b - \frac{n-1}{2} \right) \right).$$

As a direct consequence, we get the following.

Corollary A.2. *When $b = \frac{n-1}{2}$, we have $\mathbb{S}(n, \frac{n-1}{2}, \hbar) = 0$. In particular, the identity (18) holds.*

Proof of Theorem A.1. Using the equality:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \quad \text{for } 1 \leq k \leq n-1.$$

We have

$$\begin{aligned}
\mathbb{S}(n, b, \hbar) &= \sum_{\sigma \in \mathfrak{S}_n} \sigma \cdot \left(\sum_{p=0}^{n-1} (-1)^p \binom{n-1}{p} \prod_{i=1}^p (\lambda_i - b\hbar) \prod_{j=p+1}^{n-1} (\lambda_j + b\hbar) (\lambda_n + b\hbar) \prod_{1 \leq i < j \leq n} \frac{\lambda_{ij} + \hbar}{\lambda_{ji}} \right) \\
&\quad - \sum_{\sigma \in \mathfrak{S}_n} \sigma \cdot \left((\lambda_1 - b\hbar) \sum_{p=0}^{n-1} (-1)^p \binom{n-1}{p} \prod_{i=2}^{p+1} (\lambda_i - b\hbar) \prod_{j=p+2}^n (\lambda_j + b\hbar) \prod_{1 \leq i < j \leq n} \frac{\lambda_{ij} + \hbar}{\lambda_{ji}} \right) \\
&= \sum_{\sigma \in \text{Sh}_{([1, n-1], n)}} \sigma \cdot \left(\mathbb{S}(n-1, b, \hbar) (\lambda_n + b\hbar) \prod_{1 \leq i \leq n-1} \frac{\lambda_{in} + \hbar}{\lambda_{ni}} \right) \\
&\quad - \sum_{\sigma \in \text{Sh}_{(1, [2, n])}} \sigma \cdot \left(\mathbb{S}(n-1, b, \hbar) (\lambda_1 - b\hbar) \prod_{2 \leq j \leq n} \frac{\lambda_{1j} + \hbar}{\lambda_{j1}} \right) \\
(21) \quad &= \mathbb{S}(n-1, b, \hbar) \left(\sum_{j=1}^n \left((\lambda_j + b\hbar) \prod_{\{i: 1 \leq i \leq n, i \neq j\}} \frac{\lambda_{ij} + \hbar}{\lambda_{ji}} \right) - \sum_{j=1}^n \left((\lambda_j - b\hbar) \prod_{\{i: 1 \leq i \leq n, i \neq j\}} \frac{\lambda_{ji} + \hbar}{\lambda_{ij}} \right) \right).
\end{aligned}$$

Here the last equality follows from the induction hypothesis that $\mathbb{S}(n-1, b, \hbar)$ only depends on $n-1, b$ and \hbar .

It remains to compute the right hand side of (21). Define $F(t, b, \hbar) := \frac{1}{\hbar}(t + b\hbar) \prod_{i=1}^n \frac{\lambda_i - t + \hbar}{t - \lambda_i}$. The function $F(t, b, \hbar)$ has only simple poles at $t = \lambda_j$, for $j = 1, \dots, n$, and

$$\sum_{j=1}^n \text{Res}_{t=\lambda_j} F(t, b, \hbar) = \sum_{j=1}^n \left((\lambda_j + b\hbar) \prod_{\{i: 1 \leq i \leq n, i \neq j\}} \frac{\lambda_{ij} + \hbar}{\lambda_{ji}} \right).$$

By the residue theorem, $\sum_{j=1}^n \text{Res}_{t=\lambda_j} F(t, b, \hbar)$ is equal to

$$- \text{Res}_{t=\infty} F(t, b, \hbar) = - \text{Res}_{t=\infty} \frac{1}{\hbar} (t + b\hbar) \prod_{i=1}^n \left(-1 + \frac{\hbar}{t - \lambda_i} \right) = -(-1)^n \left(nb\hbar + \sum_{i=1}^n \lambda_i - \hbar \binom{n}{2} \right).$$

Therefore

$$(22) \quad \sum_{j=1}^n \left((\lambda_j + b\hbar) \prod_{\{i: 1 \leq i \leq n, i \neq j\}} \frac{\lambda_{ij} + \hbar}{\lambda_{ji}} \right) = -(-1)^n \left(nb\hbar + \sum_{i=1}^n \lambda_i - \hbar \binom{n}{2} \right).$$

Similarly, we also have

$$(23) \quad \sum_{j=1}^n \left((\lambda_j - b\hbar) \prod_{\{i: 1 \leq i \leq n, i \neq j\}} \frac{\lambda_{ji} + \hbar}{\lambda_{ij}} \right) = -(-1)^n \left(-nb\hbar + \sum_{i=1}^n \lambda_i + \hbar \binom{n}{2} \right).$$

Plugging the equalities (22) and (23) into (21), we get:

$$\begin{aligned}
\mathbb{S}(n, b, \hbar) &= \mathbb{S}(n-1, b, \hbar) \left(-(-1)^n \left(nb\hbar + \sum_{i=1}^n \lambda_i - \hbar \binom{n}{2} \right) + (-1)^n \left(-nb\hbar + \sum_{i=1}^n \lambda_i + \hbar \binom{n}{2} \right) \right) \\
&= \mathbb{S}(n-1, b, \hbar) (-1)^{n+1} 2\hbar n \left(b - \frac{n-1}{2} \right).
\end{aligned}$$

This completes the proof. \square

REFERENCES

- [BHLW14] A. Beliakova, K. Habiro, A. Lauda, and B. Webster, *Current algebras and categorified quantum groups*, preprint, (2014). [arXiv:1412.1417](#) [0.4](#)
- [CZZ14] B. Calmès, K. Zainoulline, and C. Zhong, *Equivariant oriented cohomology of flag varieties*, preprint, (2014). [arXiv:1409.7111](#) [1.2](#)
- [CG] N. Chriss, V. Ginzburg, *Representation theory and complex geometry*. Reprint of the 1997 edition. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2010. x+495 pp. [MR2838836](#) [1.5](#), [5.2](#)
- [Da15] B. Davison, *Three-dimensional symplectic geometry*, Research Program in Topology of Moduli Spaces and Representation Theory, Park City Mathematics Institute, (2015). [0.6](#)
- [Des09] D. Deshpande, *Algebraic cobordism of classifying spaces*, preprint, (2009). [arXiv:0907.4437v1](#) [1.2](#)
- [EG98] D. Edidin and W. Graham, *Equivariant intersection theory*, Invent. Math., 131(3), 595–634, (1998). [MR1614555](#) [1.2](#), [1.3](#), [1.9](#)
- [En00] B. Enriquez, *On correlation functions of Drinfeld currents and shuffle algebras*, Transform. Groups **5** (2000), 111–120. [MR1762114](#) [0](#), [0.4](#)
- [FO97] B. Feigin, A. Odesskii, *A family of elliptic algebras*. Int. Math. Res. Notices, 1997(11), 531–539. [MR1448336](#) [0.3](#), [3.3](#), [3.7](#)
- [FT09] B. Feigin and A. Tsymbaliuk, *Equivariant K-theory of Hilbert schemes via shuffle algebra*. Kyoto J. Math. **51** (2011), no. 4, 831–854. [MR2854154](#) [0](#), [0.3](#), [3.7](#)
- [Fed94] G. Felder, *Elliptic quantum groups*. XIth International Congress of Mathematical Physics (Paris, 1994), 211–218, Int. Press, Cambridge, MA, 1995. [MR1370676](#) [0](#), [0.5](#)
- [Ful84] W. Fulton, *Intersection theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 2. Springer-Verlag, Berlin, 1984. xi+470 pp. [MR0732620](#) [0.4](#)
- [GTL10] S. Gautam, V. Toledano Laredo, *Yangians and quantum loop algebras*. Selecta Mathematica **19** (2013) no. **2**, 271–336. [arXiv:1012.3687](#). [0.5](#), [7.1](#), [2](#)
- [GTL13] S. Gautam, V. Toledano Laredo, *Yangians, quantum loop algebras and abelian difference equations*, To appear in JAMS. [arXiv:1310.7318](#). [0.5](#)
- [GTL14] S. Gautam, V. Toledano Laredo, *Meromorphic Kazhdan-Lusztig equivalence for Yangians and quantum loop algebras*, preprint, [arXiv:1403.5251](#). [0.5](#)
- [GTL15] S. Gautam, V. Toledano Laredo, *Elliptic quantum groups and their finite-dimensional representations*, in preparation. [0.5](#)
- [Gin06] V. Ginzburg, *Calabi-Yau algebras*. Preprint, (2006). [arXiv:0612139](#). [0.6](#), [5.1](#)
- [Gin09] V. Ginzburg, *Lectures on Nakajima's Quiver Varieties*. Preprint, (2009). [arXiv:0905.0686](#) [0.6](#), [5.1](#)
- [GKV95] V. Ginzburg, M. Kapranov, and E. Vasserot, *Elliptic algebras and equivariant elliptic cohomology*, Preprint, (1995). [arXiv:9505012](#) [0](#), [0.5](#)
- [Gr94a] I. Grojnowski, *Delocalized equivariant elliptic cohomology*, Elliptic cohomology, London Math. Soc. Lecture Note Ser., **342**, Cambridge Univ. Press, (2007), 111–113. [MR2330509](#) [0.5](#)
- [Gr94b] I. Grojnowski, *Affinizing quantum algebras: From D-modules to K-theory*, preprint, (1994). [0.1](#), [0.5](#), [5.16](#)
- [HM13] J. Heller and J. Malagón-López, *Equivariant algebraic cobordism*, J. Reine Angew. Math., 684, 87–112, (2013). [MR3181557](#) [1.2](#)
- [HMSZ12] A. Hoffnung, J. Malagón-López, A. Savage, and K. Zainoulline, *Formal Hecke algebras and algebraic oriented cohomology theories*, Selecta Math. (N.S.) **20** (2014), no. **4**, 1247–1248. [MR3273635](#) [0.5](#)
- [KR11] A. Kleshchev and A. Ram, *Representations of Khovanov-Lauda-Rouquier algebras and combinatorics of Lyndon words*, Math. Ann. **349** (2011), 943–975. [0.4](#)
- [KoSo11] M. Kontsevich, Y. Soibelman, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants*, Commun. Number Theory Phys. **5** (2011), no. **2**, 231–352. [MR2851153](#) [0.1](#), [0.6](#), [2](#), [2.1](#), [2.1](#), [2.2](#)
- [Kr12] A. Krishna, *Equivariant cobordism of schemes*, Doc. Math., **17**, 95–134, (2012). [MR2889745](#) [1.2](#)
- [Lec04] B. Leclerc, *Dual canonical bases, quantum shuffles and q-characters*, Mathematische Zeitschrift **246** (2004), no. **4**, 691–732. [0](#), [0.4](#)
- [LM07] M. Levine, F. Morel, *Algebraic cobordism theory*, Springer, Berlin, 2007. [MR2286826](#) [0](#), [1.1](#), [1.1](#), [1.13](#)
- [LYZ13] M. Levine, Y. Yang, and G. Zhao, *Algebraic elliptic cohomology theory and flops I*, preprint, 2013. [arXiv:1311.2159](#) [1.3](#)
- [L91] G. Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras*. J. Amer. Math. Soc. **4** (1991), no. **2**, 365–421. [MR1088333](#) [0.4](#), [3.2](#), [4.1](#)

- [MO12] D. Maulik, A. Okounkov, *Quantum groups and quantum cohomology*, preprint. [arXiv:1211.1287](#) 0.4, 0.6, 7.2, 7.5, 7.2, 7.3.2
- [M13] M. McBreen, *Quantum cohomology of hypertoric varieties and geometric representations of Yangians*. Thesis (Ph.D.)—Princeton University. 2013. 77 pp. ISBN: 978-1303-45649-7 [MR3192991](#) 0.4, 7.3.1
- [Nak94] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. **76** (1994), 365–416. [MR1302318](#) 0, 5.1
- [Nak98] H. Nakajima, *Quiver varieties and Kac-Moody algebras*, Duke. Math. J., **91**, 1998, 515–560. [MR1604167](#) 5.3, 7.3.3
- [Nak01] H. Nakajima, *Quiver varieties and finite dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. **14** (2001), no. 1, 145–238. [MR1808477](#) [arXiv:9912158](#) 0, 0.2, 0.4, 0.5, 3.2, 5.3, 5.5, 5.5, 5.6, 6.2.2
- [Nak12] H. Nakajima, *Quiver varieties and tensor products II*, Symmetries, integrable systems and representations, 403–428, Springer Proc. Math. Stat., 40, Springer, Heidelberg, 2013. [MR3077693](#) 7.1
- [Ne15] A. Negut, *Quantum algebras and cyclic quiver varieties*, Ph.D. Thesis, Columbia University, (2015). 0, 0.4, 3.3
- [PPR08] I. Panin, K. Pimenov and O. Röndigs, *A universality theorem for Voevodsky’s algebraic cobordism spectrum*, Homology, Homotopy and Applications, vol. **10**(1), 2008, pp.1–16. [MR2475610](#) 1.2
- [PS04] I. Panin, A. Smirnov, *Homotopy properties of algebraic vector bundles*, Zap. Sem. POMI 319 (2004), Vopr. Teor. Predst. Algebr i Grupp 11 (2004) 261–263 (in Russian). 1.3
- [RS15] J. Ren and Y. Soibelman, *Cohomological Hall algebras, semicanonical bases and Donaldson-Thomas invariants for 2-dimensional Calabi-Yau categories*, Preprint, (2015). [arXiv:1508.06068](#) 0.6
- [Rin90] C. Ringel, *Hall algebras and quantum groups*. Invent. Math. **101** (1990), no. 3, 583–591. [MR1062796](#) 0, 0.4
- [R89] M. Rosso, *An analogue of the PBW theorem and the universal R-matrix for $U_h(\mathfrak{sl}(n+1))$* , Commun. Math. Phys. **124** (1989), 307–18. [MR1012870](#) 0
- [R98] M. Rosso, *Quantum groups and quantum shuffles*, Invent. math. **133**, 399–416 (1998). [MR1632802](#) 0, 0.4
- [SV10] O. Schiffmann, E. Vasserot, *Hall algebras of curves, commuting varieties and Langlands duality*. Math. Ann. **353** (2012), no. 4, 1399–1451. [MR2944034](#) 0.6
- [SV12] O. Schiffmann, E. Vasserot, *The elliptic Hall algebra and the K-theory of the Hilbert scheme of \mathbb{A}^2* . Duke Math. J. **162** (2013), no. 2, 279–366. [MR3018956](#) (document), 0, 0.1, 0.6, 1.4, 1.10, 1.11, 1.15, 3.2, 5.2
- [SVV14] P. Shan, M. Varagnolo, and E. Vasserot. *On the center of quiver-Hecke algebras*. Preprint, (2014). [arXiv:1411.4392](#) 0.4
- [T99] B. Totaro, *The Chow ring of a classifying space*. Algebraic K-theory (Seattle, WA, 1997), 249–281, Proc. Sympos. Pure Math., 67, Amer. Math. Soc., Providence, RI, 1999. [MR1743244](#) 1.2
- [T93] R. Thomason, *Les K-groupes d’un schéma éclaté et une formule d’intersection excédentaire*, Invent. Math. **112** (1993), 195–215. [MR1207482](#) 1.2, 1.9
- [Va00] M. Varagnolo, *Quiver varieties and Yangians*, Lett. Math. Phys. **53** (2000), no. 4, 273–283. [MR1818101](#) 0, 0.2, 0.4, 2, 7.1, 1, 7.3, 7.3.1, 7.8
- [Vi07] A. Vishik, *Symmetric operations in algebraic cobordisms*, Adv. Math. **213** (2007), no. 2, 489–552. [MR2332601](#) 1.8
- [YZ16] Y. Yang and G. Zhao, *On two cohomological Hall algebras*, preprint, (2016). [arXiv:1604.01477](#) 0.6
- [YZ1] Y. Yang and G. Zhao, *Cohomology Hall algebras and affine quantum groups*, in preparation. 0, 0.3, 0.5
- [YZ2] Y. Yang and G. Zhao, *Quiver varieties and elliptic quantum groups*, in preparation. 0, 0.5
- [ZZ14] G. Zhao and C. Zhong, *Geometric representations of the formal affine Hecke algebra*, Preprint, (2014). [arXiv:1406.1283](#) 0.5, 1.2
- [ZZ15] G. Zhao and C. Zhong, *Elliptic affine Hecke algebra and its representations*, Preprint, (2015). [arXiv:1507.01245](#) 0.5

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MASSACHUSETTS, AMHERST, MA, 01003, USA

E-mail address: yaping@math.umass.edu

MAX-PLANCK-INSTITUT FÜR MATHEMATIK, VIVATSGASSE 7, 53111 BONN, GERMANY

Current address: Institut de Mathématiques de Jussieu, UMR 7586 du CNRS, Batiment Sophie Germain, 75205 Paris Cedex 13, France

E-mail address: gufangzhao@zju.edu.cn